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Topic: Stabilizability & Detectability

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1 Review

There exists a transformation of coordinates to the system represented by

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \underbrace{\begin{bmatrix}A_{\mathtt{o}} & 0\\ A_{21} & A_{\mathtt{uo}}\end{bmatrix}}_{T^{-1}AT}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}B_{\mathtt{o}}\\ 0\end{bmatrix}}_{T^{-1}B}u\\ y &= \underbrace{\begin{bmatrix}C_{\mathtt{o}} & 0\end{bmatrix}}_{CT}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{aligned}$$

where $(x_1, x_2) \in \mathbb{C}^{n-r} \times \mathbb{C}^r$. This is the observable decomposition. This follows from the second matrix representation theorem and the fact that $\ker(\mathcal{O})$ is A-invariant and $\ker(C) \subset \|\rceil\nabla(\mathcal{O})$.

Analogously, the controllable decomposition is given by

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}A_{\mathtt{c}} & A_{12}\\ 0 & A_{\mathtt{uc}}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + B_{\mathtt{c}}u\\ y &= \begin{bmatrix}C_{\mathtt{c}} & C_{\mathtt{uc}}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{aligned}$$

where $(x_1, x_2) \in \mathcal{C}^q \times \mathbb{C}^{n-q}$. This follows from the fact that by the definition of the controllability matrix $\mathbb{R}(\mathcal{C}) \supset \mathbb{R}(B)$, hence $\mathbb{R}(B)$ is in the subspace generated by the first q basis vectors. Given the controllable decomposition, we have the following observations:

- we denote the controllable states as x_1 and uncontrollable states as x_2;
- the matrices $A_{\mathtt{c}}$ and $A_{\mathtt{uc}}$ correspond to the dynamics of the controllable and uncontrollable states, respectively;
- the matrix $B_{\mathtt{c}}$ corresponds to the coefficient of the control input to the controllable states;
- and, finally, the matrices $C_{\mathtt{uc}}$ and $C_{\mathtt{c}}$ correspond to the transformed output matrix C to these new coordinates.

The details on how to construct this representation follow from results in [510], and is described in §**??**.

1.1 Stabilizability

Definition 1. The pair (A, B) is stabilizable if there is a similarity transform to the form (**??**) with $A_{\mathtt{u}}$ Hurwitz stable.

Theorem 2. The following are equivalent:

1. The continuous-time LTI system (A, B) is stabilizable
2. Every eigenvector of $A^\top$ corresponding to an eigenvalue with a positive or zero real part is not in the kernel of $B^\top$.
3. (PBH test) $\text{rank}([A - \lambda I \;\; B]) = n$ for all $\lambda \in \mathbb{C}$ such that $\text{Re}(\lambda) \geq 0$.

4. There is a positive definite solution $P = P^\top \succ 0$ to the Lyapunov matrix inequality

$$AP + PA^\top - BB^\top < 0$$

Controller Synthesis. Like with controllability we can leverage the Lyapunov test for stabilizability in item 3 above to synthesize stabilizing feedback controllers.

Consider

$$\dot{x} = Ax + Bu$$

and suppose this system is stabilizable (i.e. all unstable modes are in the controllable subspace). Let $K := \frac{1}{2}B^\top P^{-1}$ where $P = P^\top \succ 0$ solve the Lyapunov matrix inequality

$$AP + PA^\top - BB^\top < 0$$

This inequality can be rewritten as

$$(A - \frac{1}{2}BB^\top P^{-1})P + P(A - \frac{1}{2}BB^\top P^{-1})^\top = (A - BK)P + P(A - BK)^\top < 0$$

Multiplying this equation on both sides by $Q := P^{-1}$, we obtain

$$Q(A - BK) + (A - BK)^\top Q < 0$$

so that since $Q \succ 0$, by the Lyapunov stability theorem $A - BK$ is Hurwitz stable. This in turn means that the controller $u = -Kx$ asymptotically stabilizes the system (A, B).

1.2 Detectability

Definition 3. A pair (A, C) is detectable if it is similar to a system in the standard form (**??**) with A_{uo} a Hurwitz matrix.

The above definition is stating that all unobservable modes are stable.

Theorem 4 (Detectability Tests)**.** The following are equivalent:

1. The continuous-time LTI system (A, C) is detectable
2. Every eigenvector of A corresponding to an eigenvalue with a positive or zero real part is not in the kernel of C.
3. (PBH test)
$$\text{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = n, \quad \forall \lambda \in \mathbb{C} : \ \text{Re}(\lambda) \geq 0.$$
4. There is a positive definite solution $P = P^\top \succ 0$ to the Lyapunov matrix inequality

$$AP + PA^\top - C^\top C < 0$$

Observer Synthesis. Analogous to the synthesis of stabilizing feedback, we can also use the tools in this module to synthesis observers. This amounts to designing a state estimation scheme. Consider the continuous time system

$$\dot{x} = Ax + Bu, \ y = Cx + Du$$

and let $u = -Kx$ be a stabilizing feedback controller. When only the output y can be measured, the control law cannot be implemented, but if the pair (C, A) is detectable, it should be possible to estimate x from the system's output up to an error that vanishes as $t \to \infty$.

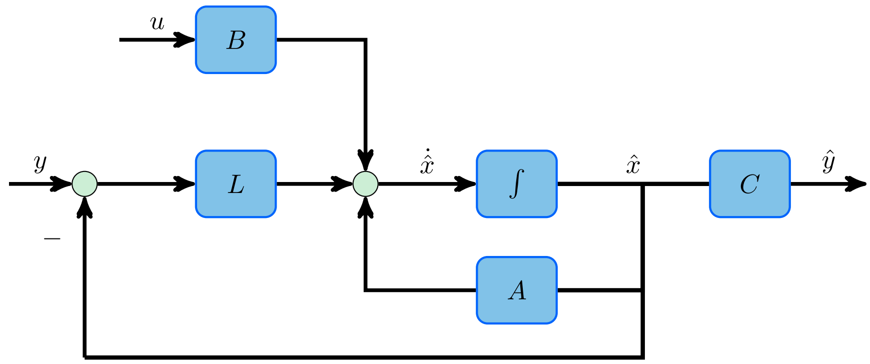

Figure 1: Observer Detailed Block Diagram

We have already seen that for an observable system, the state can be recovered from the input and output over an interval $[t_0, t_1]$ using the observability Grammian. This just gives the value at a particular time. What we want to do is design a method of recovering the state for all time.

An *observer* is a signal reconstruction device which provides an estimate of inaccessible (aka unobservable) states.

There are several ways to derive the state equations for the full-state observer. The approach in these notes is to model the observer state equations as a model of the actual system plus a correction term based on the measured output and the estimate of what that output is expected to be.

Original system and observer:

$$\begin{aligned}\dot{x} &= Ax + Bu,\\ y &= Cx\\ \dot{\hat{x}} &= (A - LC)\hat{x} + Bu + Ly\\ \hat{y} &= C\hat{x}\end{aligned}$$

where $L \in \mathbb{R}^{n\times p}$.

We call

$$e(t) = x(t) - \hat{x}(t)$$

the estimation error which satisfies

$$\dot{e} \quad = \quad (A - LC)e$$

It therefore follows that if we can choose the feedback matrix L to be such that the system matrix $(A - LC)$ has negative real parts, then

$$\hat{x} \to x, \text{ as } t \to \infty$$

(i.e. an *asymptotic estimate*) irrespective of the plant input u!

As we have already seen with pole placement, the gain matrix L of the full-state observer can be computed using any of the methods used to compute the control gain matrix K. We will assume that the system is completely observable. Therefore, the closed-loop eigenvalues of the observer can be placed at specified locations through the choice of L. For the control problem with full-state feedback, the closed-loop system matrix of interest is $A - BK$. Comparing that with the observer problem, the closed-loop system matrix is $A - LC$. The structure of those two matrices is similar; only the order of the unknown matrix differs between BK and LC.

Recall from [510] that the eigenvalues of a matrix and its transpose are the same. Hence, the observer problem can be formulated the same way as the control problem by considering the matrix $(A - LC)^\top = A^\top - C^\top L^\top$.

1.3 Problems

Problem 1. (Lyapynov Test for Stabilizability.) Show that the system (A, B) is stabilizable if and only if there exists $P = P^\top \succ 0$ such that

$$AP + PA^\top - BB^\top \prec 0$$

Solution.

Problem 2. (Stabilizability.) Consider the LTI system

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

We say a *mode* (aka eigenvalue) λ_k is a hidden uncontrollable mode if an only if

$$\text{rank}\begin{bmatrix}\lambda_k I - A & | & B\end{bmatrix} < n$$

We say such a system is stabilizable iff there are no unstable uncontrollable hidden modes—that is, any hidden uncontrollable mode λ_k must be in the open left half plane $\lambda_k \in \mathbb{C}_-^\circ$. Prove or disprove the following statement:

$$(A, B) \text{ stabilizable} \iff \{A^* v = \lambda v,\ v \neq 0 \implies B^* v \neq 0\}\ \forall\ \lambda \in \mathbb{C}_+$$

where $\mathcal{C}_+$ is the closed right half plane.

Solution.

Problem 3. (Observer Design.) Consider the system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix} x(t)\end{aligned}$$

Design an observer to place the poles of the observer at $\{-4, -4\}$.

Solution.