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Topic: Stabilizability & Detectability

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1 Review

There exists a transformation of coordinates to the system represented by

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} A_{\mathsf{o}} & 0 \\ A_{21} & A_{\mathsf{uo}} \end{bmatrix}}_{T^{-1}AT} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} B_{\mathsf{o}} \\ 0 \end{bmatrix}}_{T^{-1}B} u \\
y &= \underbrace{\begin{bmatrix} C_{\mathsf{o}} & 0 \end{bmatrix}}_{CT} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

where $(x_1, x_2) \in \mathbb{C}^{n-r} \times \mathbb{C}^r$. This is the observable decomposition. This follows from the second matrix representation theorem and the fact that $\ker(\mathcal{O})$ is A-invariant and $\ker(C) \subset \|\rceil\nabla(\mathcal{O})$.

Analogously, the controllable decomposition is given by

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{\mathsf{c}} & A_{12} \\ 0 & A_{\mathsf{uc}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B_{\mathsf{c}} u \\
y &= \begin{bmatrix} C_{\mathsf{c}} & C_{\mathsf{uc}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

where $(x_1, x_2) \in \mathcal{C}^q \times \mathbb{C}^{n-q}$. This follows from the fact that by the definition of the controllability matrix $\mathbb{R}(\mathcal{C}) \supset \mathbb{R}(B)$, hence $\mathbb{R}(B)$ is in the subspace generated by the first q basis vectors. Given the controllable decomposition, we have the following observations:

- we denote the controllable states as x_1 and uncontrollable states as x_2;
- the matrices A_{c} and A_{uc} correspond to the dynamics of the controllable and uncontrollable states, respectively;
- the matrix B_{c} corresponds to the coefficient of the control input to the controllable states;
- and, finally, the matrices C_{uc} and C_{c} correspond to the transformed output matrix C to these new coordinates.

The details on how to construct this representation follow from results in **[510]**.

1.1 Stabilizability

Definition 1. The pair (A, B) is stabilizable if there is a similarity transform to the form

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{\mathsf{c}} & A_{12} \\ 0 & A_{\mathsf{uc}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B_{\mathsf{c}} u \\
y &= \begin{bmatrix} C_{\mathsf{c}} & C_{\mathsf{uc}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

with A_{u} Hurwitz stable.

Theorem 2. The following are equivalent:

1. The continuous-time LTI system (A, B) is stabilizable
2. Every eigenvector of $A^\top$ corresponding to an eigenvalue with a positive or zero real part is not in the kernel of $B^\top$.
3. (PBH test) $\mathrm{rank}([A - \lambda I \;\; B]) = n$ for all $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) \geq 0$.
4. There is a positive definite solution $P = P^\top \succ 0$ to the Lyapunov matrix inequality
$$AP + PA^\top - BB^\top < 0$$

Controller Synthesis. Like with controllability we can leverage the Lyapunov test for stabilizability in item 3 above to synthesize stabilizing feedback controllers.

Consider
$$\dot{x} = Ax + Bu$$
and suppose this system is stabilizable (i.e. all unstable modes are in the controllable subspace). Let $K := \frac{1}{2}B^\top P^{-1}$ where $P = P^\top \succ 0$ solve the Lyapunov matrix inequality
$$AP + PA^\top - BB^\top < 0$$
This inequality can be rewritten as
$$(A - \frac{1}{2}BB^\top P^{-1})P + P(A - \frac{1}{2}BB^\top P^{-1})^\top = (A - BK)P + P(A - BK)^\top < 0$$
Multiplying this equation on both sides by $Q := P^{-1}$, we obtain
$$Q(A - BK) + (A - BK)^\top Q < 0$$
so that since $Q \succ 0$, by the Lyapunov stability theorem $A - BK$ is Hurwitz stable. This in turn means that the controller $u = -Kx$ asymptotically stabilizes the system (A, B).

1.2 Detectability

Definition 3. A pair (A, C) is detectable if it is similar to a system in the standard form
$$\begin{aligned}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} &= \underbrace{\begin{bmatrix}A_\circ & 0 \\ A_{21} & A_{\mathtt{uo}}\end{bmatrix}}_{T^{-1}AT}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}B_\circ \\ 0\end{bmatrix}}_{T^{-1}B}u \\ y &= \underbrace{\begin{bmatrix}C_\circ & 0\end{bmatrix}}_{CT}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}\end{aligned}$$
with $A_{\mathtt{uo}}$ a Hurwitz matrix.

The above definition is stating that all unobservable modes are stable.

Theorem 4 (Detectability Tests)**.** The following are equivalent:

1. The continuous-time LTI system (A, C) is detectable
2. Every eigenvector of A corresponding to an eigenvalue with a positive or zero real part is not in the kernel of C.
3. (PBH test)
$$\mathrm{rank}\left(\begin{bmatrix}A - \lambda I \\ C\end{bmatrix}\right) = n, \quad \forall \lambda \in \mathbb{C} : \; \mathrm{Re}(\lambda) \geq 0.$$
4. There is a positive definite solution $P = P^\top \succ 0$ to the Lyapunov matrix inequality
$$AP + PA^\top - C^\top C < 0$$

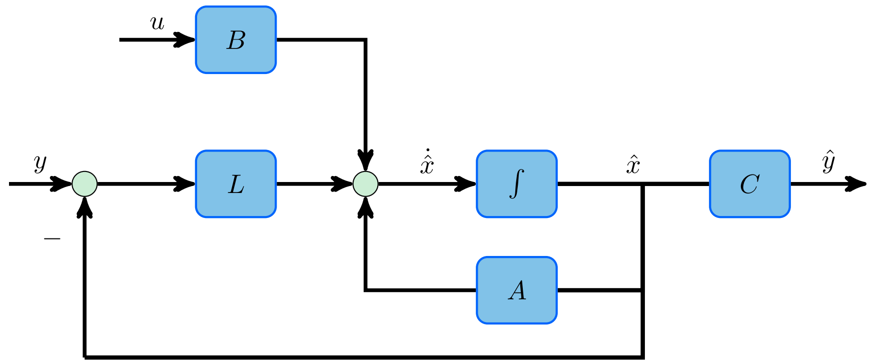

Figure 1: Observer Detailed Block Diagram

Observer Synthesis. Analogous to the synthesis of stabilizing feedback, we can also use the tools in this module to synthesis observers. This amounts to designing a state estimation scheme. Consider the continuous time system

$$\dot{x} = Ax + Bu, \ \ y = Cx + Du$$

and let $u = -Kx$ be a stabilizing feedback controller. When only the output y can be measured, the control law cannot be implemented, but if the pair (C, A) is detectable, it should be possible to estimate x from the system's output up to an error that vanishes as $t \to \infty$.

We have already seen that for an observable system, the state can be recovered from the input and output over an interval $[t_0, t_1]$ using the observability Grammian. This just gives the value at a particular time. What we want to do is design a method of recovering the state for all time.

An *observer* is a signal reconstruction device which provides an estimate of inaccessible (aka unobservable) states.

There are several ways to derive the state equations for the full-state observer. The approach in these notes is to model the observer state equations as a model of the actual system plus a correction term based on the measured output and the estimate of what that output is expected to be.

Original system and observer:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx \\ \dot{\hat{x}} &= (A - LC)\hat{x} + Bu + Ly \\ \hat{y} &= C\hat{x} \end{aligned}$$

where $L \in \mathbb{R}^{n \times p}$.

We call

$$e(t) = x(t) - \hat{x}(t)$$

the estimation error which satisfies

$$\dot{e} \ \ = \ \ (A - LC)e$$

It therefore follows that if we can choose the feedback matrix L to be such that the system matrix $(A - LC)$ has negative real parts, then

$$\hat{x} \to x, \ \text{as } t \to \infty$$

(i.e. an *asymptotic estimate*) irrespective of the plant input u!

As we have already seen with pole placement, the gain matrix L of the full-state observer can be computed using any of the methods used to compute the control gain matrix K. We will assume that the system is completely observable. Therefore, the closed-loop eigenvalues of the observer can be placed at specified locations through the choice of L. For the control problem with full-state feedback, the closed-loop system matrix of interest is $A - BK$. Comparing that with the observer problem, the closed-loop system matrix is $A - LC$. The structure of those two matrices is similar; only the order of the unknown matrix differs between BK and LC.

Recall from **[510]** that the eigenvalues of a matrix and its transpose are the same. Hence, the observer problem can be formulated the same way as the control problem by considering the matrix $(A - LC)^\top = A^\top - C^\top L^\top$.

1.3 Problems

Problem 1. (Lyapynov Test for Stabilizability.) Show that the system (A, B) is stabilizable if and only if there exists $P = P^\top \succ 0$ such that

$$AP + PA^\top - BB^\top \prec 0$$

Solution. We first prove directly by contradiction that a solution to the matrix inequality exists implies that (A, B) is stabilizable. The simplest way to do this is by using the eigenvector test. Assume that

$$AP + PA^\top - BB^\top \prec 0$$

holds, and let $x \neq 0$ be an eigenvector of $A^\top$ associated with the "unstable" eigenvalue λ; i.e., $A^\top x = \lambda x$. Then

$$x^*(AP + PA^\top)x < x^* BB^\top x = \|B^\top x\|^2$$

But the left hand side of this is equal to

$$(A^\top x^*)^\top Px + x^* PA^\top x = \lambda^* x^* Px + \lambda x^* Px = 2\mathrm{Re}(\lambda) x^* Px$$

Since $P = P^\top \succ 0$ and $\mathrm{Re}(\lambda) \geq 0$ we have that

$$0 \leq 2\mathrm{Re}(\lambda) x^* Px < \|B^\top x\|^2$$

Therefor $x \notin \mathrm{Ker}(B)$ and hence the system is stabilizable.

For the other direction, we assume (A, B) is stabilizable. Let T be the similarity transform that takes the system to the controllable decomposition:

$$\bar{A} := \begin{bmatrix} A_{\mathsf{c}} & A_{12} \\ 0 & A_{\mathsf{uc}} \end{bmatrix} = T^{-1}AT \quad \text{and} \quad \bar{B} := \begin{bmatrix} B_{\mathsf{c}} \\ 0 \end{bmatrix} = T^{-1}B$$

We saw in Section 12.4 (regarding feedback stabilization based on the Lyapunov test) that controllability of the pair $(A_{\mathsf{c}}, B_{\mathsf{c}})$ guarantees the existence of a positive-definite matrix P_c such that

$$A_{\mathsf{c}} P_c + P_c A_{\mathsf{c}}^\top - B_{\mathsf{c}} B_{\mathsf{c}}^\top = -Q_c \prec 0$$

On the other hand, since A_{uc} is a stability matrix, we conclude from the Lyapunov stability theorem that there exists a positive- definite matrix P_u such that

$$A_{\mathsf{uc}} P_u + P_u A_{\mathsf{uc}}^\top = -Q_u \prec 0.$$

Define

$$\bar{P} = \mathrm{blkdiag}(P_c, \rho P_u)$$

for some scalar $\rho > 0$ to be determined in a min. Then we have that

$$\begin{aligned}
&\bar{A}\bar{P} + \bar{P}\bar{A}^\top - \bar{B}\bar{B}^\top \\
&= \begin{bmatrix} A_{\mathsf{c}} & A_{12} \\ 0 & A_{\mathsf{uc}} \end{bmatrix} \mathrm{blkdiag}(P_c, \rho P_u) + \mathrm{blkdiag}(P_c, \rho P_u) \begin{bmatrix} A_{\mathsf{c}} & A_{12} \\ 0 & A_{\mathsf{uc}} \end{bmatrix}^\top - \begin{bmatrix} B_{\mathsf{c}} \\ 0 \end{bmatrix} \begin{bmatrix} B_{\mathsf{c}} \\ 0 \end{bmatrix}^\top \\
&= -\begin{bmatrix} Q_c & -\rho A_{12} P_u \\ -\rho P_u A_{12}^\top & \rho Q_u \end{bmatrix}
\end{aligned}$$

It turns out that by making ρ positive, but sufficiently small, the right-hand side can be made negative-definite. The proof is completed by verifying that

$$P = T \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix} T^\top$$

satisfies the matrix inequality.

Problem 2. (Stabilizability.) Consider the LTI system

$$\begin{aligned}
\dot{x} &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned}$$

We say a *mode* (aka eigenvalue) λ_k is a hidden uncontrollable mode if an only if

$$\mathrm{rank} \begin{bmatrix} \lambda_k I - A & | & B \end{bmatrix} < n$$

We say such a system is stabilizable iff there are no unstable uncontrollable hidden modes—that is, any hidden uncontrollable mode λ_k must be in the open left half plane $\lambda_k \in \mathbb{C}_-^\circ$. Prove or disprove the following statement:

$$(A, B) \text{ stabilizable} \iff \{A^* v = \lambda v,\ v \neq 0 \implies B^* v \neq 0\}\ \forall\ \lambda \in \mathbb{C}_+$$

where $\mathcal{C}_+$ is the closed right half plane.

Solution. ($\Longrightarrow$): suppose (A, B) is stabilizable, but there exists a vector $v = 0$ such that $A^* v = \lambda v$, $B^* v = 0$. This implies that

$$v^* A = \bar{\lambda} v^*,\ v^* B = 0$$

Hence,

$$v^*(A + BF) = \bar{\lambda} v^*\ \forall F$$

Since $A_{cl} = A + BF$ has an eigenvalue in $\bar{\mathbb{C}}_+$, this contradicts that the realization is stabilizable.

($\Longleftarrow$): Suppose

$$\{A^* v = \lambda v,\ v \neq 0 \implies B^* v \neq 0\}\ \forall\ \lambda \in \mathbb{C}_+$$

but (A, B) not stabilizable. We can put the realization in kalman decomposition form:

$$A + BF = \begin{bmatrix} A_{11} + B_1 F & * \\ 0 & A_{22} \end{bmatrix}$$

where A_{22} is not stable (since (A, B) not stabilizable). Then there exists $w \neq 0$ such that $w^* A_{22} = \lambda w^*$, $\lambda \in \bar{\mathbb{C}}_+$. We get that

$$v^*(A + BF) = \begin{bmatrix} 0 & w^* \end{bmatrix} (A + BF) = \begin{bmatrix} 0 & w^* \end{bmatrix} \begin{bmatrix} A_{11} + B_1 F & * \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & \lambda w^* \end{bmatrix} = \lambda v^*,\ \forall F$$

Choosing $F = 0$, we obtain $v^* A = \lambda v^*$ and

$$v^* B = \begin{bmatrix} 0 & w^* \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = 0$$

Thus, we have found a vector $v \neq 0$ where $B^* v = 0$ which is a contradiction.

Problem 3. (Observer Design.) Consider the system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix} x(t) \end{aligned}$$

Design an observer to place the poles of the observer at $\{-4, -4\}$.

Solution. It is easy to check that the system is completely observable. Let $L = (\ell_1, \ell_2)$ be the unknown observer gain. Write the generic state estimation matrix

$$A - LC = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2}\ell_1 \\ 1 & -1 - \frac{1}{2}\ell_2 \end{bmatrix}$$

The characteristic polynomial of the observer is

$$\det(\lambda I - A + LC) = \lambda^2 + \left(2 + \frac{1}{2}\ell_2\right)\lambda + \frac{1}{2}\ell_2 + \frac{1}{2}\ell_1 + 1$$

Impose the polynomial equals the desired one

$$(\lambda + 4)^2 = \lambda^2 + 8\lambda + 16$$

Then we solve the linear system of equations in ℓ_1, ℓ_2 to get

$$\ell_1 = 18, \quad \ell_2 = 12$$

The resulting Luenberger observer is

$$\frac{d\hat{x}}{dt} = \begin{bmatrix} -1 & -9 \\ 1 & -7 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 18 \\ 12 \end{bmatrix} y(t)$$

Stabilization Through Output Feedback. Consider again the following LTI system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

that is asymptotically stabilized by the state feedback control law

$$u = -Kx$$

and let

$$\dot{\hat{x}} = A\hat{x} + Bu - L(\hat{y} - y), \quad \hat{y} = C\hat{x}$$

be a state estimator for which $A - LC$ is a stability matrix. If the state x cannot be measured, one may be tempted to use the state estimate $\hat{x}$ instead of the actual state x in the control—i.e,

$$u = -K\hat{x}$$

This results in a controller with the following state-space model

$$\begin{aligned} &\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y), \quad u = -K\hat{x} \\ \iff &\dot{\hat{x}} = (A - LC - BK)\hat{x} + Ly \quad u = -K\hat{x} \end{aligned}$$

To study whether or not the closed loop system is stable we recall that

$$\dot{e} = (A - LC)e$$

and

$$\dot{x} = Ax + Bu = (A - BK)x - BKe$$

Putting these together we have the dynamcis

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

The following theorem results from the triangular structure of this matrix.

Theorem 5. The closed loop of the LTI system with the output feedback controller $u = -K\hat{x}$ results in a system whose eigenvalues are the union of the eigenvalues of the state feedback closed-loop matrix $A - BK$ with the eigenvalues of the state estimator matrix $A - LC$.

This is called separation of estimation and control.