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Topic: Midterm Review

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1 Solutions to LTV/LTI Systems

Consider the following LTV system:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \quad \text{(state DE)} \\ y(t) &= C(t)x(t) + D(t)u(t) \quad \text{(read-out/output eqn.)} \end{aligned}$$

with initial data (t_0, x_0) and the assumptions on $A(\cdot), B(\cdot), C(\cdot), D(\cdot), u(\cdot)$ all being piecewise continuous (PC):

- $A(t) \in \mathbb{R}^{n\times n}$
- $B(t) \in \mathbb{R}^{n\times m}$
- $C(t) \in \mathbb{R}^{p\times n}$
- $D(t) \in \mathbb{R}^{p\times m}$

The input function $u(\cdot) \in \mathcal{U}$, where $\mathcal{U}$ is the set of piecewise continuous functions from $\mathbb{R}_+ \to \mathbb{R}^m$.

This system satisfies the assumptions of our existence and uniqueness theorem. Indeed,

1. For all fixed $x \in \mathbb{R}^n$, the function $t \in \mathbb{R}_+\backslash\mathcal{D} \to f(x,t) \in \mathbb{R}^n$ is continuous where $\mathcal{D}$ contains all the points of discontinuity of $A(\cdot), B(\cdot)$, $C(\cdot)$, $D(\cdot)$, $u(\cdot)$
2. There is a PC function $k(\cdot) = \|A(\cdot)\|$ such that

$$\|f(\xi,t) - f(\xi',t)\| = \|A(t)(\xi-\xi')\| \le k(t)\|\xi-\xi'\| \quad \forall t \in \mathbb{R}_+,\ \forall \xi,\xi' \in \mathbb{R}^n$$

Hence, by the above theorem, the differential equation has a unique continuous solution $x : \mathbb{R}_+ \to \mathbb{R}^n$ which is clearly defined by the parameters $(t_0, x_0, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times U$.

Theorem 1. (Existence of the state transition map/flow.) Under the assumptions and notation above, for every triple $(t_0, x_0, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times U$, the state transition map

$$x(\cdot) = \phi(\cdot, t_0, x_0, u) : \mathbb{R}_+ \to \mathbb{R}^n$$

is a continuous map well-defined as the unique solution of the state differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

with (t_0, x_0) such that $x(t_0) = x_0$ and $u(\cdot) \in U$.

The solution to the LTV system is given by

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\ d\tau$$

where $\Phi(t,t_0)$ is the state transition matrix. Note that the state transition matrix satisfies the differential equation

$$\dot{X} = A(t)X, \quad X(t_0) = I.$$

Continuous time LTI Systems. Consider now the general LTI system in state-space form:

$$\dot{x} = Ax + Bu \tag{1}$$
$$y = Cx + Du \tag{2}$$

where

- $x \in \mathbb{R}^n$ is the "state" of the system
- $u \in \mathbb{R}^m$ is the "input" to the system
- $y \in \mathbb{R}^p$ is the "output" of the system
- $A \in \mathbb{R}^{n\times n}$ describes how the state changes in time (dynamics)
- $B \in \mathbb{R}^{n\times m}$ describes how the input effects the state dynamics
- $C \in \mathbb{R}^{p\times n}$ describes how the state is transformed to the output
- $D \in \mathbb{R}^{p\times m}$ describes how the input is transformed to the output (for the most part in this class we take $D = 0$).

The solution to the CT LTI system in (1) is

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\ d\tau$$

Discrete time LTI Systems. A discrete time LTI system is given by

$$x[k+1] = Ax[k] + Bu[k] \tag{3}$$
$$y[k] = Cx[k] + Du[k] \tag{4}$$

The solution for the DT LTI system is given by

$$x[k] = A^{k-k_0}x[k_0] + \sum_{\ell=k_0}^{k-1} A^{k-\ell-1}Bu[\ell]$$

2 Stability of Linear Systems

Definition 2 (Stable Equilibrium)**.** The following are characterizations of stability (in the sense of Lyapunov).

a. **Marginally Stable**: Consider the equilibrium point $x^* = 0$.

$$x^* \text{ is stable } \iff \ \forall\ x_0 \in \mathbb{R}^n,\ \forall t_0 \in \mathbb{R}^n,\ t \mapsto x(t) = \Phi(t,t_0)x_0 \text{ is bounded } \forall\ t \geq t_0.$$

Note: the effect of initial conditions does not grow unbounded with time (but it may grow temporarily during a transient phase).

b. **Asymptotic Stability**. Consider the equilibrium point $x^* = 0$.

$$x^* = 0 \text{ is asymptotically stable } \iff\ x_0 = 0 \text{ is stable and } x(t) = \Phi(t,t_0)x_0 \longrightarrow 0 \text{ as } t \to \infty.$$

Note: the effect of initial conditions eventually disappears with time.

c. **Exponential Stability**. Consider the equilibrium point $x^* = 0$.

$$x^* = 0 \text{ is exponentially stable } \iff \exists M, \alpha > 0 :\ \|x(t)\| \leq M\exp(-\alpha(t-t_0))\|x_0\|$$

2.1 Spectral Conditions for Stability

Proposition 3 (Continuous Time)**.** Consider the differential equation $\dot{x} = Ax$, $x(0) = x_0$. From the above expression:

$$\{\exp(At) \to 0 \ \text{ as } \ t \to \infty\} \iff \{\forall \lambda_k \in \mathrm{spec}(A),\ \mathrm{Re}(\lambda_k) < 0\}$$

and

$$\{t \mapsto \exp(At) \text{ is bounded on } \mathbb{R}_+\} \iff \left\{ \begin{array}{ll} \forall \lambda_k \in \mathrm{spec}(A), & \mathrm{Re}(\lambda_k) \leq 0 \ \& \\ m_k = 1 \text{ when} & \mathrm{Re}(\lambda_k) = 0 \end{array} \right\}$$

Claim 1.

$$\dot{x} = Ax \text{ is exponentially stable } \iff \mathrm{spec}(A) \subset \mathbb{C}_-^\circ$$

Linearized System Stability. Consider a general non-linear system

$$\dot{x} = f(x),\ x \in \mathbb{R}^n$$

with an equilibrium point x^* such that $f(x^*) = 0$. Recall that the local linearization around x^* is given by

$$\dot{\tilde{x}} = A\tilde{x}$$

with $\tilde{x} = x - x^*$ and $A := Df(x^*)$. The following theorem is the celebrated Hartman-Grobman theorem which states that trajectories of the nonlinear system are "equivalent" to trajectories of the linearization in a neighborhood of an equilibrium, and hence we can assess (local) stability of the nonlinear system by assessing stability of the linearized system.[1]

Theorem 4 (Hartman-Grobman)**.** Consider a nonlinear dynamical system $\dot{x} = f(x)$ with an equilibrium point x^* (i.e. $f(x^*) = 0$). If the linearization of the system $A := D_x f(x)|_{x=x^*}$ has no zero or purely imaginary eigenvalues then there exists a homeomorphism (i.e., a continuous map with a continuous inverse) from a neighborhood U of x^* into $\mathbb{R}^n$,

$$h : U \to \mathbb{R}^n,$$

taking trajectories of the nonlinear system $\dot{x} = f(x)$ and mapping them onto those of $\dot{\tilde{x}} = A\tilde{x}$. In particular, we have that x^* maps to the equilibrium of the linearized system—i.e., $h(x^*) = 0$.

The above theorem directly translates to the following corollary.

Corollary 5. Suppose that $f \in C^2(\mathbb{R}^n, \mathbb{R})$. If the linearized system is exponentially stable, then there exists a ball $B \subset \mathbb{R}^n$ around x^* and constants $c, \lambda > 0$ such that for every solution $x(t)$ to the nonlinear system that starts at $x(t_0) \in B$, we have

$$\|x(t) - x^*\| \leq ce^{-\lambda(t-t_0)}\|x(t_0) - x^*\|.$$

This means that the properties of the linearized system are preserved in the nonlinear system.

Numerical Integration. Spectral stability properties are also important for numerical integration schemes such as forward and backward Euler. In particular choosing the step size for the integration scheme determines the stability of the discrete time update equation, and the choice of step size depends on the eigenvalues of A.

[1] If you are interested in learning more about nonlinear systems, I suggest Shankar Sastry's book "Nonlinear Systems" [sastry2013nonlinear].

2.2 Lyapunov Stability

Consider a general dynamical system

$$\dot{x} = f(x)$$

Without loss of generality we will discuss critical points at $x = 0$. Recall the definitions of stability:

- **stable**: an equilibrium $x = 0$ is stable if for all $t_0 \geq 0$ and $\epsilon > 0$, there exists $\delta(t_0, \epsilon)$ such that

$$\|x_0\| < \delta(t_0, \epsilon) \implies \|x(t)\| < \epsilon, \quad \forall t \geq t_0$$

- **uniformly stable**: an equilibrium $x = 0$ is uniformly stable if δ can be chosen independent of t_0
- **asymptotically stable**: an equilibrium point $x^* = 0$ of $\dot{x} = f(x)$ is said to be asymptotically stable if for every trajectory $x(t)$ we have that $x(t) \to 0$ as $t \to \infty$

Beyond the spectral conditions we saw last time for linear systems and local linearizations for nonlinear systems (by way of Hartman Grobman), another method to check for stability is to construct a function (namely, a Lyapunov function) which acts as a certificate for stability.

Theorem 6 (Lyapynov Theorem)**.** Consider the dynamical system defined by $f \in C^1(\mathbb{R}^n, \mathbb{R})$. Let W be an open subset of $\mathbb{R}^n$ containing the equilibrium point x^*—i.e., $f(x^*) = 0$. Suppose that there exists a real-valued function $V \in C^1$ such that $V(x^*) = 0$ and $V(x) > 0$ when $x \neq x^*$. Then, the following implications hold:

a. $\dot{V}(x) \leq 0, \forall x \in W \implies x^*$ is stable.
b. $\dot{V}(x) < 0, \forall x \in W \backslash \{x^*\} \implies x^*$ is asymptotically stable.
c. $\dot{V}(x) > 0, \forall x \in W \backslash \{x^*\} \implies x^*$ is unstable.

For linear systems, it turns out that Lyapunov functions take the form

$$V(z) = z^\top P z$$

for some positive definite symmetric matrix $P \succ 0$.

For a linear system $\dot{x} = Ax$, if

$$V(z) = z^\top P z$$

then if the system is stable, we will have

$$\dot{V}(z) = (Az)^\top P z + z^\top P(Az) = z^\top (A^\top P + PA) z < 0$$

This means that for the system to be asymptotically stable, we want it to be the case that for any $Q = Q^\top \succ 0$, there exists a $P = P^\top \succ 0$ that solves

$$A^\top P + PA = -Q$$

If $P \succ 0$, then the sublevel sets[2] of this function are ellipsoids and bounded. Further, we have that

$$V(z) = z^\top P z = 0 \iff z = 0.$$

If $P \succ 0, Q \succeq 0$, then all the trajectories of $\dot{x} = Ax$ are bounded (i.e., $\mathrm{Re}(\lambda_i) \leq 0$ and if $\mathrm{Re}(\lambda_i) = 0$, then λ_i corresponds to a Jordan block of size one). Further, the ellipsoids $\{z |\ z^\top P z \leq a\}$ are invariant sets.

Moreover, if we think of $z^\top P z$ as the (generalized) energy, then $z^\top Q z$ is the associated (generalized) dissipation.

[2] i.e., $\{x |\ V(x) < a\}$

Theorem 7. The following conditions are equivalent:

a. The system $\dot{x} = Ax$ is asymptotically (equivalently exponentially) stable.
b. All the eigenvalues of A have strictly negative real parts.
c. For every symmetric positive definite matrix $Q = Q^\top \succ 0$, there exists a unique solution P to the *Lyapunov equation*

$$A^\top P + PA = -Q.$$

Moreover, P is symmetric and positive-definite—i.e., $P = P^\top \succ 0$—and is given by

$$P = \int_0^\infty e^{A^\top t} Q e^{At} \, dt.$$

d. There exists a symmetric positive-definite matrix $P = P^\top \succ 0$ for which the following Lyapunov matrix inequality holds:

$$A^\top P + PA < 0$$

Why is Lyapunov useful? The Lyapunov equation allows us to synthesize controllers (as well as observers) that induce the system to be stable.

3 Controllability & Observability

- Controllability/Reachability is the property of a system concerning the ability to steer the state from arbitrary x_0 to arbitrary x_1 on a given time interval $[t_0, t_1]$.
- Observability is the property of a system concerning the ability to uniquely recover the initial state x_0 given the observation y.

3.1 Controllability/Reachability

For simplicity, let $D(\cdot) \equiv 0$ and consider our state space to be $\mathbb{R}^n$.

Definition 8 (Controllable)**.** The system $\mathcal{D} = (A(\cdot), B(\cdot), C(\cdot))$ is controllable on $[t_0, t_1]$ if for all $(x_0, x_1) \in \mathbb{R}^n$, there exists $u_{[t_0,t_1]} \in \mathcal{U}$ which steers x_0 at t_0 to x_1 at t_1.

We tend to break controllability into two different concepts:

- Controllability from the origin (reachability): the reachability map is given by

$$\mathcal{L}_r u(t) = \int_{t_0}^{t} \Phi(t, \tau) B(\tau) u(\tau) \, d\tau$$

  If this map is surjective (i.e., $\text{Im}(\mathcal{L}_r) = \mathbb{R}^n$) then the system is reachable.
- Controllability to the origin (often simply referred to as contorllability, hence, one should take care to identify the precise definition in the given reference they are looking at): the controllability map is given by

$$\mathcal{L}_c u(t) = \int_{t_0}^{t} \Phi(t_0, \tau) B(\tau) u(\tau) \, d\tau$$

  If this map is surjective (i.e., $\text{Im}(\mathcal{L}_c) = \mathbb{R}^n$) then the system is controllable.

We can compute the adjoint of these two maps $\mathcal{L}_r^*$ and $\mathcal{L}_c^*$ respectively, and define the corresponding grammians

$$\mathcal{L}_r\mathcal{L}_r^* = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^*(\tau)\Phi(t_1,\tau)^* \; d\tau$$

and

$$\mathcal{L}_c\mathcal{L}_c^* = \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B^*(\tau)\Phi(t_0,\tau)^* \; d\tau$$

These are both linear operators and hence by the matrix representation theorem there is a finite dimension matrix representation $W_r \in \mathbb{R}^{n\times n}$ and $W_c \in \mathbb{R}^{n\times n}$, respectively. The finite rank operator lemma let's us show that

$$\mathrm{Im}(\mathcal{L}_r) = \mathrm{Im}(\mathcal{L}_r\mathcal{L}_r^*) = \mathrm{Im}(W_r)$$

and

$$\mathrm{Im}(\mathcal{L}_c) = \mathrm{Im}(\mathcal{L}_c\mathcal{L}_c^*) = \mathrm{Im}(W_c)$$

Moreover, we have the following equivalences:

$$\begin{aligned} (A(\cdot), B(\cdot)) \;\text{ controllable on } [t_0,t_1] &\iff \mathrm{Im}(\mathcal{L}_r) = \mathbb{R}^n \\ &\iff \mathrm{Im}(\mathcal{L}_c) = \mathbb{R}^n \\ &\iff \mathrm{Im}(\mathcal{L}_r\mathcal{L}_r^*) = \mathbb{R}^n \\ &\iff \mathrm{Im}(\mathcal{L}_c\mathcal{L}_c^*) = \mathbb{R}^n \\ &\iff \det(W_r) \neq 0 \\ &\iff \det(W_c) \neq 0 \end{aligned}$$

LTI Systems. Things get much easier in the LTI setting. Cayley Hamilton allows us to show that the condition on the controllability (reachability) grammian can be essentially reduced to checking a rank condition on the so called controllability matrix:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times nm}$$

Fact 9. The following equality holds:

$$\mathrm{Im}(W_r) = \mathrm{Im}(\mathcal{C})$$

Let $\Delta = t_1 - t_0$ for some $t_1 > t_0$.

Theorem 10. The following are equivalent:

$$\begin{aligned} &\text{The LTI system is completely controllable on some } [0,\Delta] \\ \iff &\operatorname{rank}\left(\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}\right) = n && \text{(Rank Test)} \\ \iff &\operatorname{rank}\left(\begin{bmatrix} sI - A & B \end{bmatrix}\right) = n, \;\; \forall\, s \in \mathbb{C} && \text{(PBH Test)} \end{aligned}$$

3.2 Observability

The pair $(A(\cdot), C(\cdot))$ is observable if given output $y(t)$, the initial state x_0 can be uniquely recovered.

Definition 11. The state x_0 is unobservable on $[t_0, t_1]$ if and only if its zero input response is zero on $[t_0, t_1]$.

Analogous to controllability, we can define the observablility map $\mathcal{L}_o : \mathbb{R}^n \to \mathcal{Y}_{[t_0,t_1]}$ by

$$\mathcal{L}_o x_0(\cdot) = C(\cdot)\Phi(\cdot, t_0)x_0$$

That is, $\mathcal{L}_\circ x_0$ is an operator in $\mathrm{PC}([t_0,t_1]))$ such that

$$(\mathcal{L}_\circ x_0)(t) = y(t) = \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\ d\tau$$

We have the following equivalences:

$$\begin{aligned}(A(\cdot),C(\cdot)) \text{ is completely observable (CO) on } [t_0,t_1] \quad &\iff \quad \mathrm{Ker}(\mathcal{L}_\circ) = \{0\}\\ &\iff \quad \mathrm{Ker}(\mathcal{L}_\circ^*\mathcal{L}_\circ) = \{0\}\\ &\iff \quad \det(W_\circ) \neq 0\end{aligned}$$

where

$$W_\circ = \int_{t_0}^{t_1} \Phi(\tau,t_0)^* C(\tau)^* C(\tau)\Phi(\tau,t_0)\ d\tau$$

LTI Systems. Just as with controllability, we can use Cayley Hamilton to construct an observability matrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Theorem 12 (LTI Observability Tests)**.** The following are equivalent:

$$\text{The LTI system is completely observable on some } [0,\Delta]$$

$$\iff \quad \mathrm{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n \qquad \text{(Rank Test)}$$

$$\iff \quad \mathrm{rank}\left(\begin{bmatrix} sI - A \\ C \end{bmatrix}\right) = n, \quad \forall\ s \in \mathbb{C} \qquad \text{(PBH Test)}$$

Problem 1. (Lipschitz and ODEs.) Consider the following two systems of differential equations:

$$\text{(a)} \qquad \begin{cases} \dot{x}_1 &= -x_1 + e^t \cos(x_1 - x_2) \\ \dot{x}_2 &= -x_2 + 15 \sin(x_1 - x_2) \end{cases}$$

and

$$\text{(b)} \qquad \begin{cases} \dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= -x_2 \end{cases}$$

1. Does (a) satisfy a global Lipschitz condition?
2. Does (b) satisfy a global Lipschitz condition?
3. For (b), your friend from MIT asserts that the solutions are uniquely defined for all possible initial conditions and they all tend to zero for all initial conditions. Do you agree or disagree?

Solution.

1. Does (a) satisfy a global Lipschitz condition?

   For the first system, we have
   $$f(x,t) = \begin{pmatrix} -x_1 + e^t \cos(x_1 - x_2) \\ -x_2 + 15 \sin(x_1 - x_2) \end{pmatrix}$$
   We can now apply Fact 3.1 from the discussion notes using the norm $\|\cdot\|_\infty$ (since all norms on finite–dimensional vector spaces are equivalent, we can select the most convenient one). Then, we have
   $$\|f(x,t) - f(y,t)\| \le \|Df\| \|x - y\| \le \max\{1 + 2e^t, 31\} \|x - y\|$$

2. Does (b) satisfy a global Lipschitz condition?
   For the second system, we claim that it is not globally Lipschitz. Indeed, we only need to find an x and y such that there is no constant k such that
   $$\|f(x) - f(y)\| \le k\|x - y\|$$
   where
   $$f(x) = \begin{pmatrix} -x_1 + x_1 x_2 \\ -x_2 \end{pmatrix}$$
   Consider
   $$x = \begin{pmatrix} a \\ a \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
   Then,
   $$\left| f\begin{pmatrix} a \\ a \end{pmatrix} - f\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right| = \left| \begin{pmatrix} x(-1+x) \\ x \end{pmatrix} \right| = \left| \begin{pmatrix} x \\ x \end{pmatrix} \right| \left| \begin{pmatrix} x-1 \\ 1 \end{pmatrix} \right| > k \left| \begin{pmatrix} x \\ x \end{pmatrix} \right| = k \left| \begin{pmatrix} x \\ x \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right|$$
   For any $k > 0$, we can find a vector $\begin{pmatrix} a \\ a \end{pmatrix}$ such that the above holds.

3. For the second system, your friend asserts that the solutions are uniquely defined for all possible initial conditions and they all tend to zero for all initial conditions. Do you agree or disagree?

   Agree. The second ODE for x_2 does not depend on x_1 and it satisfies the conditions of the ODE theorem. Hence, it has a solution $x_2(t) = e^{-t} x_2(0)$. We can plug this into the ODE $\dot{x}_1 = -x_1 + x_1 x_2$ for x_2. Then, we can see that the resulting ODE satisfies the conditions of the ODE theorem. Hence, it has a solution given by $x_1(t) = x_1(0) \exp(-t + x_2(0)(\exp(-t) - 1))$. It is easy to see that $x_i(t) \to 0$ as $t \to \infty$ for all initial conditions and for each $i \in \{1, 2\}$.

□

Problem 2. (Observability Map and Differential Equation). Recall that the observability map is given by

$$\mathcal{L}_o : x_0 \to C(\cdot)\Phi(\cdot, t)x_0$$

and

$$\mathcal{L}_o^* y(\cdot) = \int_{t_0}^{t_1} \Phi^*(\tau, t_0)C^*(\tau)y(\tau)\ d\tau$$

Show that $\mathcal{L}_o^*\mathcal{L}_o$ satisfies

$$\dot{X}(t) = -A(t)^* X(t) - X(t)A(t) - C^*(t)C(t),\ X(t_1) = 0$$

Solution. The trick is to think of $\mathcal{L}_o^*\mathcal{L}_o$ as an operator where t_0 is the argument.

Claim:

$$W_o[t, t_1] = W_o[t_1](t) = \mathcal{L}_o^*\mathcal{L}_o(t) = \int_t^{t_1} \Phi^*(\tau, t)C^*(\tau)C(\tau)\Phi(\tau, t)\ d\tau$$

satisfies the given differential equation.
Indeed, recall problem 24.2 in which we showed that

$$\frac{d}{d\tau}\Phi(t, \tau) = -\Phi(t, \tau)A(\tau)$$

Then, using Leibniz rule,

$$\begin{aligned}
\frac{d}{dt}\left(\mathcal{L}_o^*\mathcal{L}_o(t)\right) &= \cancelto{0}{\frac{d}{dt}(t_1)}\Phi^*(t_1, t)C^*(t_1)C(t_1)\Phi(t_1, t) - \cancelto{1}{\frac{d}{dt}(t)}\Phi^*(t, t)C^*(t)C(t)\Phi(t, t) \\
&\quad + \int_t^{t_1} \frac{d}{dt}\left(\Phi^*(\tau, t)C^*(\tau)C(\tau)\Phi(\tau, t)\right)\ d\tau \\
&= -C^*(t)C(t) - \int_t^{t_1} \Big(A^*(t)\Phi^*(\tau, t)C^*(\tau)C(\tau)\Phi(\tau, t) \\
&\quad + \Phi^*(\tau, t)C^*(\tau)C(\tau)\Phi(\tau, t)A(t)\Big)\ d\tau \\
&= -C^*(t)C(t) - A^*(t)\left(\int_t^{t_1} \Phi^*(\tau, t)C^*(\tau)C(\tau)\Phi(\tau, t)\ d\tau\right) \\
&\quad - \left(\int_t^{t_1} \Phi^*(\tau, t)C^*(\tau)C(\tau)\Phi(\tau, t)\ d\tau\right) A(t)
\end{aligned}$$

Lastly, we must check the initial condition. Indeed,

$$\mathcal{L}_o^*\mathcal{L}_o(t_1) = \int_{t_1}^{t_1} \Phi^*(\tau, t)C^*(\tau)C(\tau)\Phi(\tau, t)\ d\tau = 0$$

□

Problem 3. (LTV Lyapunov.) We can actually construct a similar quadratic Lyapunov function for LTV systems. Prove the following theorem.

Theorem 13. Assume that $A(\cdot)$ is bounded. If for some $Q(t) > \alpha I$,

$$P(t) = \int_t^{\infty} \Phi^\top(\tau, t)Q(\tau)\Phi(\tau, t)\ d\tau$$

is bounded, then the origin is a uniformly asymptotically stable equilibrium point of $\dot{x} = A(t)x$.

Solution.

Claim 2. If $Q(t) > 0$ for all $t \geq 0$ and

$$\alpha \|x\|^2 \leq x^\top Q(t) x \quad \forall\ x,\ t \geq 0,$$

and $A(\cdot)$ is bounded by k, then $P(t)$ is uniformly positive definite—i.e., $\exists \beta$ such that

$$\beta \|x\|^2 \leq x^\top P(t) x$$

Proof. First observe that

$$x^\top P(t) x = x^\top \left(\int_t^\infty \Phi^\top(\tau,t) Q(\tau) \Phi(\tau,t)\ d\tau \right) x \geq \alpha \int_t^\infty \|\Phi(\tau,t)x\|^2 d\tau \geq \alpha \int_t^\infty \|x\|^2 e^{-2k(\tau-t)} d\tau$$

where the last inequality holds by Bellman-Grownwall which gives a lower bound on $\|\Phi(\tau,t)x\|$ from the bound $\|A(t)\| \leq k$, as

$$\|\Phi(\tau,t)x\| \geq \|x\| e^{-k(\tau-t)}$$

Computing the integral we have that

$$x^\top P(t) x \geq \frac{\alpha}{2k} \|x\|^2$$

so that $\beta := \alpha/(2k)$. □

Now we can prove the theorem. By assumption $P(t)$ is bounded—i.e., there exists some $\gamma > 0$ such that

$$x^\top P(t) x \leq \gamma \|x\|^2$$

The preceding claim gives us a lower bound. Hence $v(x,t) = x^\top P(t) x$ is a positive definite function. Now, we have that

$$\dot{v} = \dot{x}^\top P(t) x + x^\top \dot{P}(t) x + x^\top P(t) \dot{x}$$

We need an expression for $\dot{P}$. This follows from direct computation: first recall that we showed earlier in the quarter that

$$\frac{d}{dt} \Phi(\tau,t) = -\Phi(\tau,t) A(t)$$

Hence

$$\begin{aligned}
\dot{P}(t) &= \frac{d}{dt} \left(\int_t^\infty \Phi^\top(\tau,t) Q(\tau) \Phi(\tau,t)\ d\tau \right) \\
&= -Q(t) + \int_t^\infty \dot{\Phi}^\top(\tau,t) Q(\tau) \Phi(\tau,t) + \Phi^\top(\tau,t) Q(\tau) \dot{\Phi}(\tau,t)\ d\tau \\
&= -Q(t) - \int_t^\infty A^\top \Phi^\top(\tau,t) Q(\tau) \Phi(\tau,t)\ d\tau - \int_t^\infty \Phi^\top(\tau,t) Q(\tau) \Phi(\tau,t) A(t)\ d\tau \\
&= -Q(t) - A^\top(t) P(t) - P(t) A(t)
\end{aligned}$$

Then we deduce that

$$\begin{aligned}
\dot{v} &= \dot{x}^\top P(t) x + x^\top \dot{P}(t) x + x^\top P(t) \dot{x} \\
&= x^\top A^\top(t) P(t) x + x^\top (-Q(t) - A^\top(t) P(t) - P(t) A(t)) x + x^\top P(t) A(t) x \\
&= -x^\top Q(t) x \\
&\leq -\alpha \|x\|^2
\end{aligned}$$

Hence, v is a Lyapunov function and the origin is exponentially stable.

Problem 4. (LTI Observability/Controllability.) Consider the LTI system given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x$$

Is the system controllable? Observable?

Solution. The system is in controllability form, so its controllable. However we can verify this by checking the rank of the controllability matrix

$$\text{rank}\mathcal{C} = \text{rank} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 3 \end{bmatrix} = 3$$

For observability, let's check the observability matrix

$$\text{rank}\mathcal{O} = \text{rank} \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -2 \\ 3 & 5 & 8 \end{bmatrix} = 3$$

Problem 5. (LTV Controllability.) Derive a matrix differential equation that

$$t \mapsto W_c[t, t_1] = \int_t^{t_1} \Phi(t,\tau)B(\tau)B^*(\tau)\Phi(t,\tau)\ d\tau$$

solves.

Problem 6. (Pole Placement.) Consider the dynamic system

$$\frac{d^4\theta}{dt^4} + \alpha_1 \frac{d^3\theta}{dt^3} + \alpha_2 \frac{d^2\theta}{dt^2} + \alpha_3 \frac{d\theta}{dt} + \alpha_4 \theta = u$$

where u represents an input force, α_i are real scalars. Assuming that $\frac{d^3\theta}{dt^3}$, $\frac{d^2\theta}{dt^2}$, $\frac{d\theta}{dt}$ and θ can all be measured, design a state variable feedback control scheme which places the closed–loop eigenvalues at $s_1 = -1$, $s_2 = -1$, $s_3 = -1 + j1$, $s_4 = -1 - j$.

Solution. First, given the closed loop poles, the desired characteristic equation is

$$\pi(s) = (s+1)(s+1)(s+1-j)(s+1+j) = (s+1)^2(s^2+2s+2) = s^4 + 4s^3 + 7s^2 + 6s + 2$$

Let $\theta_1 = \dot{\theta}$, $\theta_2 = \ddot{\theta}$, and $\theta_3 = \frac{d^3}{dt^3}\theta = \theta^{(3)}$ (where we are using the superscript in parenthesis to denote the order of the derivative). Then

$$\dot{\theta}_3 + \alpha_1 \dot{\theta}_2 + \alpha_2 \dot{\theta}_1 + \alpha_3 \dot{\theta} + \alpha_4 \theta = u$$

so that

$$\begin{bmatrix} \dot{\theta} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_4 & -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix} \begin{bmatrix} \theta \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

Consider

$$A + bf^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_4 + f_1 & -\alpha_3 + f_2 & -\alpha_2 + f_3 & -\alpha_1 + f_4 \end{bmatrix}$$

so that to compute the characteristic polynomial of the closed loop system we do

$$\begin{aligned} \det\left(sI - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_4 + f_1 & -\alpha_3 + f_2 & -\alpha_2 + f_3 & -\alpha_1 + f_4 \end{bmatrix} \right) \\ = \det\left(\begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & s & -1 \\ \alpha_4 - f_1 & \alpha_3 - f_2 & \alpha_2 - f_3 & s + \alpha_1 - f_4 \end{bmatrix} \right) \\ = s^4 + (\alpha_1 - f_4)s^3 + (\alpha_2 - f_3)s^2 + (\alpha_3 - f_2)s + (\alpha_4 - f_1) \end{aligned}$$

(You should know how to compute this determinant when the matrix is in controllable canonical form).

Now we solve for the coefficients f_i by equating the polynomials. That is,

$$s^4 + (\alpha_1 - f_4)s^3 + (\alpha_2 - f_3)s^2 + (\alpha_3 - f_2)s + (\alpha_4 - f_1) = s^4 + 4s^3 + 7s^2 + 6s + 2$$

gives equations

$$\begin{aligned} \alpha_1 - f_4 &= 4 \\ \alpha_2 - f_3 &= 7 \\ \alpha_3 - f_2 &= 6 \\ \alpha_4 - f_1 &= 2 \end{aligned}$$

so that

$$f = \begin{bmatrix} \alpha_4 - 2 & \alpha_3 - 6 & \alpha_2 - 7 & \alpha_1 - 4 \end{bmatrix}$$

□