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Topic: Lyapunov Stability

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1 Overview: Lyapunov

Consider a general dynamical system

$$\dot{x} = f(x)$$

Without loss of generality we will discuss critical points at $x = 0$. Recall the definitions of stability:

- **stable**: an equilibrium $x = 0$ is stable if for all $t_0 \geq 0$ and $\epsilon > 0$, there exists $\delta(t_0, \epsilon)$ such that

$$\|x_0\| < \delta(t_0, \epsilon) \implies \|x(t)\| < \epsilon, \quad \forall t \geq t_0$$

- **uniformly stable**: an equilibrium $x = 0$ is uniformly stable if δ can be chosen independent of t_0
- **asymptotically stable**: an equilibrium point $x^* = 0$ of $\dot{x} = f(x)$ is said to be asymptotically stable if for every trajectory $x(t)$ we have that $x(t) \to 0$ as $t \to \infty$

Beyond the spectral conditions we saw last time for linear systems and local linearizations for nonlinear systems (by way of Hartman Grobman), another method to check for stability is to construct a function (namely, a Lyapunov function) which acts as a certificate for stability.

Theorem 1 (Lyapynov Theorem)**.** Consider the dynamical system defined by $f \in C^1(\mathbb{R}^n, \mathbb{R})$. Let W be an open subset of $\mathbb{R}^n$ containing the equilibrium point x^*—i.e., $f(x^*) = 0$. Suppose that there exists a real-valued function $V \in C^1$ such that $V(x^*) = 0$ and $V(x) > 0$ when $x \neq x^*$. Then, the following implications hold:

a. $\dot{V}(x) \leq 0, \forall x \in W \implies x^*$ is stable.
b. $\dot{V}(x) < 0, \forall x \in W \backslash \{x^*\} \implies x^*$ is asymptotically stable.
c. $\dot{V}(x) > 0, \forall x \in W \backslash \{x^*\} \implies x^*$ is unstable.

For linear systems, it turns out that Lyapunov functions take the form

$$V(z) = z^\top P z$$

for some positive definite symmetric matrix $P \succ 0$.

For a linear system $\dot{x} = Ax$, if

$$V(z) = z^\top P z$$

then if the system is stable, we will have

$$\dot{V}(z) = (Az)^\top P z + z^\top P(Az) = z^\top (A^\top P + PA) z < 0$$

This means that for the system to be asymptotically stable, we want it to be the case that for any $Q = Q^\top \succ 0$, there exists a $P = P^\top \succ 0$ that solves

$$A^\top P + PA = -Q$$

If $P \succ 0$, then the sublevel sets[1] of this function are ellipsoids and bounded. Further, we have that

$$V(z) = z^\top P z = 0 \iff z = 0.$$

[1] i.e., $\{x |\ V(x) < a\}$

If $P \succ 0, Q \succeq 0$, then all the trajectories of $\dot{x} = Ax$ are bounded (i.e., $\mathrm{Re}(\lambda_i) \leq 0$ and if $\mathrm{Re}(\lambda_i) = 0$, then λ_i corresponds to a Jordan block of size one). Further, the ellipsoids $\{z|\ z^\top Pz \leq a\}$ are invariant sets.

Moreover, if we think of $z^\top Pz$ as the (generalized) energy, then $z^\top Qz$ is the associated (generalized) dissipation.

Theorem 2. The following conditions are equivalent:

a. The system $\dot{x} = Ax$ is asymptotically (equivalently exponentially) stable.
b. All the eigenvalues of A have strictly negative real parts.
c. For every symmetric positive definite matrix $Q = Q^\top \succ 0$, there exists a unique solution P to the *Lyapunov equation*
$$A^\top P + PA = -Q.$$
Moreover, P is symmetric and positive-definite—i.e., $P = P^\top \succ 0$—and is given by
$$P = \int_0^\infty e^{A^\top t} Q e^{At}\ dt.$$
d. There exists a symmetric positive-definite matrix $P = P^\top \succ 0$ for which the following Lyapunov matrix inequality holds:
$$A^\top P + PA < 0$$

1.1 Problems

Problem 1. (Lyapunov Equation.) Consider the linear map $\mathcal{L} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $\mathcal{L}(P) = A^\top P + PA$. Show that if $\lambda_i + \bar{\lambda}_j \neq 0$, for all $\lambda_i, \lambda_j \in \sigma(A)$, the equation
$$A^\top P + PA = Q$$
has a unique symmetric solution for given symmetric Q.
Solution.

To prove this we need to show the following technical lemma.

Lemma 3. Any injective linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is also surjective.

Now towards solving the problem, we show that $\mathcal{L}(P) = Q$ has a unique solution. It suffices to show that $\mathcal{L}(P) \mapsto A^\top P + PA$ is bijective or equivalently $\mathrm{Ker}(\mathcal{L}) = \{0\}$.

Assume A is diagonalizable. What follows can be generalized even if this is not possible - the qualitative results still hold.

Lemma 4. The eigenvalues of $\mathcal{L}(P) \mapsto A^\top P + PA$ are $\lambda_i + \bar{\lambda}_j$ for all $\lambda_i, \lambda_j \in \mathrm{spec}(A)$.

What is the point of this? If A is stable (all eigenvalues have negative real parts) then $\bar{\lambda}_i + \lambda_j \neq 0$ for all combinations.

Hence, the operator $\mathcal{L}$ is linear and invertible so that we can explicitly solve the equation

$$\mathcal{L}(P) = -Q$$

and if you recall from **[Mod2-RL2]** the solution is

$$P = \int_0^\infty e^{A^*\tau} Q e^{A\tau} \, d\tau$$