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# Topic: Lyapunov Stability

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## 1 Overview: Lyapunov

Consider a general dynamical system

$$\dot{x} = f(x)$$

Without loss of generality we will discuss critical points at $x = 0$. Recall the definitions of stability:

- **stable**: an equilibrium $x = 0$ is stable if for all $t_0 \geq 0$ and $\epsilon > 0$, there exists $\delta(t_0, \epsilon)$ such that

$$\|x_0\| < \delta(t_0, \epsilon) \implies \|x(t)\| < \epsilon, \quad \forall t \geq t_0$$

- **uniformly stable**: an equilibrium $x = 0$ is uniformly stable if $\delta$ can be chosen independent of $t_0$
- **asymptotically stable**: an equilibrium point $x^* = 0$ of $\dot{x} = f(x)$ is said to be asymptotically stable if for every trajectory $x(t)$ we have that $x(t) \to 0$ as $t \to \infty$

Beyond the spectral conditions we saw last time for linear systems and local linearizations for nonlinear systems (by way of Hartman Grobman), another method to check for stability is to construct a function (namely, a Lyapunov function) which acts as a certificate for stability.

**Theorem 1** (Lyapynov Theorem)**.** Consider the dynamical system defined by $f \in C^1(\mathbb{R}^n, \mathbb{R})$. Let $W$ be an open subset of $\mathbb{R}^n$ containing the equilibrium point $x^*$—i.e., $f(x^*) = 0$. Suppose that there exists a real-valued function $V \in C^1$ such that $V(x^*) = 0$ and $V(x) > 0$ when $x \neq x^*$. Then, the following implications hold:

a. $\dot{V}(x) \leq 0, \forall x \in W \implies x^*$ is stable.
b. $\dot{V}(x) < 0, \forall x \in W \backslash \{x^*\} \implies x^*$ is asymptotically stable.
c. $\dot{V}(x) > 0, \forall x \in W \backslash \{x^*\} \implies x^*$ is unstable.

For linear systems, it turns out that Lyapunov functions take the form

$$V(z) = z^\top P z$$

for some positive definite symmetric matrix $P \succ 0$.

For a linear system $\dot{x} = Ax$, if

$$V(z) = z^\top P z$$

then if the system is stable, we will have

$$\dot{V}(z) = (Az)^\top P z + z^\top P(Az) = z^\top (A^\top P + PA) z < 0$$

This means that for the system to be asymptotically stable, we want it to be the case that for any $Q = Q^\top \succ 0$, there exists a $P = P^\top \succ 0$ that solves

$$A^\top P + PA = -Q$$

If $P \succ 0$, then the sublevel sets[1] of this function are ellipsoids and bounded. Further, we have that

$$V(z) = z^\top P z = 0 \iff z = 0.$$

[1] i.e., $\{x |\ V(x) < a\}$

If $P \succ 0, Q \succeq 0$, then all the trajectories of $\dot{x} = Ax$ are bounded (i.e., $\mathrm{Re}(\lambda_i) \leq 0$ and if $\mathrm{Re}(\lambda_i) = 0$, then $\lambda_i$ corresponds to a Jordan block of size one). Further, the ellipsoids $\{z | \; z^\top P z \leq a\}$ are invariant sets.

Moreover, if we think of $z^\top P z$ as the (generalized) energy, then $z^\top Q z$ is the associated (generalized) dissipation.

**Theorem 2.** The following conditions are equivalent:

a. The system $\dot{x} = Ax$ is asymptotically (equivalently exponentially) stable.
b. All the eigenvalues of $A$ have strictly negative real parts.
c. For every symmetric positive definite matrix $Q = Q^\top \succ 0$, there exists a unique solution $P$ to the *Lyapunov equation*
$$A^\top P + PA = -Q.$$
Moreover, $P$ is symmetric and positive-definite—i.e., $P = P^\top \succ 0$—and is given by
$$P = \int_0^\infty e^{A^\top t} Q e^{At} \, dt.$$
d. There exists a symmetric positive-definite matrix $P = P^\top \succ 0$ for which the following Lyapunov matrix inequality holds:
$$A^\top P + PA < 0$$

## 1.1 Problems

**Problem 1.** (Lyapunov Equation.) Consider the linear map $\mathcal{L} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $\mathcal{L}(P) = A^\top P + PA$. Show that if $\lambda_i + \bar{\lambda}_j \neq 0$, for all $\lambda_i, \lambda_j \in \sigma(A)$, the equation

$$A^\top P + PA = -Q$$

has a unique symmetric solution for given symmetric $Q$.
**Solution.**

To prove this we need to show the following technical lemma.

**Lemma 3.** Any injective linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is also surjective.

*Proof.* From the Rank-Nullity Theorem, we have that

$$\dim \mathrm{Im}(A) + \dim \mathrm{Ker}(A) = n$$

A map $A : U \to V$ is surjective if and only if $\dim(\mathrm{Im}(A)) = \dim V$ by definition. Here $\dim V = n$. Thus, a map is surjective if and only if $n = \dim V = n - \dim(\mathrm{Ker}(A))$. Hence, $A$ is surjective if and only if $\dim(\mathrm{Ker}(A)) = 0$ and the $\dim(\mathrm{Ker}(A)) = 0$ if and only if $\mathrm{Ker}(A) = \{0\}$ which we claim is true if and only if $A$ is injective.

Let us now prove this claim:
($\Longrightarrow$) Suppose $A$ is injective. We want to show $\mathrm{Ker}(A) = \{0\}$. First, $\{0\} \subset \mathrm{Ker}(A)$ since $A \cdot 0 = 0$. It remains to show that $\mathrm{Ker}(A) \subset \{0\}$. Let $v \in \mathrm{Ker}(A)$ so that $Av = 0$. But we also know $A \cdot 0 = 0$ so that $Av = A \cdot 0 = 0$. Injectivity of $A$ then implies that $v = 0$.
($\Longleftarrow$) On the other hand, suppose $\mathrm{Ker}(A) = \{0\}$. We want to show $A$ is injective. Indeed, suppose $Av_1 = Av_2$ for some $v_1, v_2 \in \mathbb{R}^n$. Subtracting and using linearity of $A$, we see that

$$Av_1 - Av_2 = A(v_1 - v_2) = 0$$

so that $v_1 - v_2 \in \mathrm{Ker}(A)$ but by assumption $\mathrm{Ker}(A) = \{0\}$ which, in turn, implies that $v_1 = v_2$. □

Now towards solving the problem, we show that $\mathcal{L}(P) = Q$ has a unique solution. It suffices to show that $\mathcal{L}(P) \mapsto A^\top P + PA$ is bijective or equivalently $\text{Ker}(\mathcal{L}) = \{0\}$.

Assume $A$ is diagonalizable. What follows can be generalized even if this is not possible - the qualitative results still hold.

**Lemma 4.** The eigenvalues of $\mathcal{L}(P) \mapsto A^\top P + PA$ are $\lambda_i + \bar{\lambda}_j$ for all $\lambda_i, \lambda_j \in \text{spec}(A)$.

*Proof.* Let $(\lambda_i, v_i)$ be an eigenpair for $A^\top$ so that $A^\top v_i = \lambda_i v_i$. Hence, $v_i^* A = \bar{\lambda}_i v_i^*$. Then, we have that

$$\mathcal{L}(v_j v_i^*) = A^\top v_j v_i^* + v_j v_i^* A = (\lambda_j + \bar{\lambda}_i) v_j v_i^*$$

We simply need to show that $X_{ij} = v_i v_j^*$ for all $i, j$ forms a full set of eigenmatrices for the operator $\mathcal{L}$. Let

$$V = \begin{bmatrix} | & \cdots & | \\ v_1 & \cdots & v_n \\ | & \cdots & | \end{bmatrix} \quad \text{and} \quad V^{-1} = \begin{bmatrix} -- & w_1^\top & -- \\ \vdots & \vdots & \vdots \\ -- & w_n^\top & -- \end{bmatrix}.$$

we need to simply show that the $X_{ij}$ are linearly independent. Suppose $\alpha_{ij}$ are scalars and

$$\sum_{j=1}^n \sum_{i=1}^n \alpha_{ij} X_{ij} = 0$$

Premultiply this with $w_l^\top$ and post multiply with $\bar{w}_k$ so that

$$\begin{aligned} 0 &= w_l^\top \left( \sum_{j=1}^n \sum_{i=1}^n \alpha_{ij} X_{ij} \right) w_k \\ &= \sum_{j=1}^n \sum_{i=1}^n \alpha_{ij} w_l^\top v_i v_j^* \bar{w}_k \\ &= \sum_{j=1}^n \sum_{i=1}^n \alpha_{ij} \delta_{li} \delta_{jk} \\ &= \alpha_{lk} \end{aligned}$$

which holds for any $1 \leq k, l \leq n$. Hence all the $\alpha$'s are zero so that we have a linearly indepdnent set of matrices. □

In particular, we showed that the eigenvalues of the map. Hence, if the eigenvalues are all non-zero and all such that $\lambda_i + \bar{\lambda}_j \neq 0$, then the map $\mathcal{L}$ is invertible.

We also showed in Lemma 3 that square invertible maps are also surjective and hence, bijective.

This is an alternative way to see it: Fix $P \in \mathcal{N}(\mathcal{P})$. For any eigenvector $z$ of $A$ with eigenvalue $\lambda$, we have

$$(A^\top P + PA)z = 0 \implies A^\top P z = -\lambda P z$$

Hence, either $-\lambda$ is an eigenvalue of $A^\top$ or $Pz = 0$. If $-\lambda$ is an eigenvalue of $A^\top$, then both $-\lambda, \lambda \in \sigma(A)$, which contradicts our assumption. Hence, $Pz = 0$. By the same reasoning as before, we now have that $Pv = 0$ for all generalized eigenvectors of degree 1. One can continue this until all of the eigenvectors and generalized eigenvectors of $A$ are shown to map to zero under $P$. Note that the eigenvectors and generalized eigenvectors of $A$ form a basis for $\mathbb{R}^n$, so that $P = 0$. Thus $\mathcal{N}(\mathcal{L}) = \{0\}$.

Now, we need to show that it is symmetric. Note first that $Q = Q^\top$. Hence,

$$A^\top P + PA = P^\top A + A^\top P^\top$$

which implies that $\mathcal{L}(P - P^\top) = 0$. Thus, $P = P^\top$.

**What is the point of this?** If $A$ is stable (all eigenvalues have negative real parts) then $\bar{\lambda}_i + \lambda_j \neq 0$ for all combinations.

Hence, the operator $\mathcal{L}$ is linear and invertible so that we can explicitly solve the equation

$$\mathcal{L}(P) = -Q$$

and if you recall from **[Mod2-RL2]** the solution is

$$P = \int_0^\infty e^{A^*\tau} Q e^{A\tau} \, d\tau$$

*Proof.* Suppose not, then there exists $\lambda_k, \lambda_j$ such that $\lambda_k + \bar{\lambda}_j = 0$

$$a_k + ib_k - a_j + ib_j = (a_k - a_j) + i(b_k + b_j) = 0 \iff (a_k - a_j) = -i(b_k + b_j)$$

but complex conjugates are also eigenvalues so that $\bar{\lambda}_k$ and $\lambda_j$ are also eigenvalues. Hence, taking the complex conjugate we have

$$0 = (\lambda_k + \bar{\lambda}_j)^* = \bar{\lambda}_k + \lambda_j$$

so that

$$\bar{\lambda}_k + \lambda_j = \lambda_k + \bar{\lambda}_j$$

which is equivalent to

$$a_k - ib_k + a_j + ib_j = a_k + ib_k + a_j - ib_j \iff -i(b_k - b_j) = i(b_k - b_j) \iff -i = i \;\; \rightarrow\leftarrow$$

□