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| EE/AA547-W22 | 2022.01.18 |

**Topic: Lyapunov Stability**

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# 1 Overview

**Definition 1** (Stable Equilibrium)**.** The following are characterizations of stability (in the sense of Lyapunov).

a. **Marginally Stable**: Consider the equilibrium point $x^* = 0$.

$$x^* \text{ is stable } \iff \forall\, x_0 \in \mathbb{R}^n,\ \forall t_0 \in \mathbb{R}^n,\ t \mapsto x(t) = \Phi(t, t_0)x_0 \text{ is bounded } \forall\, t \geq t_0.$$

Note: the effect of initial conditions does not grow unbounded with time (but it may grow temporarily during a transient phase).

b. **Asymptotic Stability**. Consider the equilibrium point $x^* = 0$.

$$x^* = 0 \text{ is asymptotically stable } \iff x_0 = 0 \text{ is stable and } x(t) = \Phi(t, t_0)x_0 \longrightarrow 0 \text{ as } t \to \infty.$$

Note: the effect of initial conditions eventually disappears with time.

c. **Exponential Stability**. Consider the equilibrium point $x^* = 0$.

$$x^* = 0 \text{ is exponentially stable } \iff \exists M, \alpha > 0 :\ \|x(t)\| \leq M \exp(-\alpha(t - t_0))\|x_0\|$$

**Spectral Conditions for Stability.**

**Proposition 2** (Continuous Time)**.** Consider the differential equation $\dot{x} = Ax$, $x(0) = x_0$. From the above expression:

$$\{\exp(At) \to 0 \ \text{ as } \ t \to \infty\} \iff \{\forall \lambda_k \in \operatorname{spec}(A),\ \operatorname{Re}(\lambda_k) < 0\}$$

and

$$\{t \mapsto \exp(At) \text{ is bounded on } \mathbb{R}_+\} \iff \left\{ \begin{array}{ll} \forall \lambda_k \in \operatorname{spec}(A), & \operatorname{Re}(\lambda_k) \leq 0 \ \& \\ m_k = 1 \text{ when} & \operatorname{Re}(\lambda_k) = 0 \end{array} \right\}$$

**Claim 1.**

$$\dot{x} = Ax \text{ is exponentially stable } \iff \operatorname{spec}(A) \subset \mathbb{C}_-^\circ$$

**Linearized System Stability.** Consider a general non-linear system

$$\dot{x} = f(x),\ x \in \mathbb{R}^n$$

with an equilibrium point $x^*$ such that $f(x^*) = 0$. Recall that the local linearization around $x^*$ is given by

$$\dot{\tilde{x}} = A\tilde{x}$$

with $\tilde{x} = x - x^*$ and $A := Df(x^*)$. The following theorem is the celebrated Hartman-Grobman theorem which states that trajectories of the nonlinear system are "equivalent" to trajectories of the linearization in a neighborhood of an equilibrium, and hence we can assess (local) stability of the nonlinear system by assessing stability of the linearized system.[1]

[1] If you are interested in learning more about nonlinear systems, I suggest Shankar Sastry's book "Nonlinear Systems" [sastry2013nonlinear].

**Theorem 3** (Hartman-Grobman)**.** Consider a nonlinear dynamical system $\dot{x} = f(x)$ with an equilibrium point $x^*$ (i.e. $f(x^*) = 0$). If the linearization of the system $A := D_x f(x)|_{x=x^*}$ has no zero or purely imaginary eigenvalues then there exists a homeomorphism (i.e., a continuous map with a continuous inverse) from a neighborhood $U$ of $x^*$ into $\mathbb{R}^n$,

$$h : U \to \mathbb{R}^n,$$

taking trajectories of the nonlinear system $\dot{x} = f(x)$ and mapping them onto those of $\dot{\tilde{x}} = A\tilde{x}$. In particular, we have that $x^*$ maps to the equilibrium of the linearized system—i.e., $h(x^*) = 0$.

The above theorem directly translates to the following corollary.

**Corollary 4.** Suppose that $f \in C^2(\mathbb{R}^n, \mathbb{R})$. If the linearized system is exponentially stable, then there exists a ball $B \subset \mathbb{R}^n$ around $x^*$ and constants $c, \lambda > 0$ such that for every solution $x(t)$ to the nonlinear system that starts at $x(t_0) \in B$, we have

$$\|x(t) - x^*\| \le ce^{-\lambda(t-t_0)}\|x(t_0) - x^*\|.$$

This means that the properties of the linearized system are preserved in the nonlinear system.

**Numerical Integration.** Spectral stability properties are also important for numerical integration schemes such as forward and backward Euler. In particular choosing the step size for the integration scheme determines the stability of the discrete time update equation, and the choice of step size depends on the eigenvalues of $A$.

## 2 Problems

**Problem 1.** (Stability and Eigenvalues). Consider the system

$$\dot{x} = \underbrace{\begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{=:A} x$$

Is the system asymptotically stable? Is the system stable?

**solution**. The system is stable but not asymptotically stable since all its eigenvalues are in the CLHP and the ones on the $j\omega$-axis have Jordan sub-blocks of size 1.

We can immediately see from the structure of $A$ that the eigenvalues are in the closed left-half plane and that the ones on the $j\omega$-axis have Jordan sub-blocks of size 1, so its (marginally) stable but not asymptotically stable.

Another way to see this is the following: A system is considered asymptotically stable if and only if $\Phi(t, 0) \to 0$ as $t \to \infty$ (cf. lecture notes). Using functions of matrices, we have

$$e^{At} = \begin{bmatrix} e^{-3t} & -te^{-3t} & -\frac{1}{2}t^2e^{-3t} & 0 & 0 & 0 & 0 \\ 0 & e^{-3t} & -te^{-3t} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-3t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-4t} & -te^{-4t} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-4t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Taking the limit as $t \to \infty$ and applying L'Hospital's rule, we see that

$$\lim_{t\to\infty} e^{At} = \lim_{t\to\infty} \begin{bmatrix} e^{-3t} & -te^{-3t} & -\frac{1}{2}t^2e^{-3t} & 0 & 0 & 0 & 0 \\ 0 & e^{-3t} & -te^{-3t} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-3t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-4t} & -te^{-4t} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-4t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \nrightarrow 0$$

Thus, the system is not asymptotically stable.

**Problem 2.** (Stiff Differential Equations.) In the simulation of several engineering systems we encounter parasitic elements which result in the differential equation becoming "stiff". A problem is stiff if it contains widely varying time scales, i.e., some components of the solution decay much more rapidly than others. For example, parasitic capacitances and inductances in electronic circuits. This has consequences not just in terms of the behavior of the actual system (which can experience thigns like hysterisis etc) but also for numerical simulation.

**Fact 5.** A stiff differential equation is numerically unstable unless the step size is extremely small. Stiff differential equations are characterized as those whose exact solution has a term of the form $e^{-\alpha t}$ where $\alpha$ is a large positive constant. And, large derivatives of $e^{-\alpha t}$ give error terms that are dominating in the solution.

This results in some state variables changing much more rapidly than the others. To represent this, consider the system with $x_1$ representing the "slow" variables and $x_2$ the "fast" variables given by

$$\begin{aligned} \dot{x}_1 &= \ A_{11}x_1 + A_{12}x_2 \\ \epsilon\dot{x}_2 &= \ A_{21}x_1 + A_{22}x_2 \end{aligned}$$

with $x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^m$ and $A_{22}$ non-singular.

a. Prove the following claim:

**Claim 2.** For the system above, $m$ of the eigenvalues tend to $\infty$ like $\frac{\sigma(A_{22})}{\epsilon}$, and the other $n$ eigenvalues tend to $\text{spec}(A_{11} - A_{12}A_{22}^{-1}A_{21})$.

b. The system

$$\begin{aligned} \dot{x}_1 &= \ A_{11}x_1 + A_{12}x_2 \\ 0 &= \ A_{21}x_1 + A_{22}x_2 \end{aligned}$$

is referred to as the *singularly perturbed* or *low frequency approximation*.

In electronic circuits, we also have in addition to parasitic (small) capacitances, coupling (large) capacitances. These are modeled by

$$\begin{aligned} \dot{x}_1 &= \ A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \\ \epsilon\dot{x}_2 &= \ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \\ \mu\dot{x}_3 &= \ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \end{aligned}$$

with $\epsilon > 0$ small and $\mu > 0$ large. A *mid frequency model* takes $\epsilon = 0, \mu = \infty$, a *low frequency model* takes $\epsilon = 0$ and sets $\mu = \infty$ in the $\tau = \frac{t}{\mu}$ time scale and a *high frequency model* sets $\mu = \infty$ and then sets $\epsilon = 0$ in the time scale $\tau = \frac{t}{\epsilon}$. Find the relationship between the eigenvalues of each if these models.

**Solution.**

a. *Prove the following claim:*

**Claim 3.** For the system above, $m$ of the eigenvalues tend to $\infty$ like $\frac{\sigma(A_{22})}{\epsilon}$, and the other $n$ eigenvalues tend to $\mathrm{spec}(A_{11} - A_{12}A_{22}^{-1}A_{21})$.

*Proof.* For the system

$$\begin{aligned} \dot{x}_1 =& \quad A_{11}x_1 + A_{12}x_2 \\ \varepsilon\dot{x}_2 =& \quad A_{21}x_1 + A_{22}x_2 \end{aligned}$$

as $\varepsilon \to 0$, we can approximate the second equation as

$$0 = A_{21}x_1 + A_{22}x_2,$$

so that for non–singular $A_{22}$, we have $x_2 = -A_{22}^{-1}A_{21}x_1$. Plugging this into the first differential equation in the system we get

$$\dot{x}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1$$

which has $n$ eigenvalues $\mathrm{spec}(A_{11} - A_{12}A_{22}^{-1}A_{21})$. So $n$ eigenvalues of the original system tend to these as $\varepsilon \to 0$. Call the solution to this equation $\hat{x}_1(t)$. Then we have from the second differential equation

$$\dot{x}_2 = \frac{A_{22}}{\varepsilon}x_2 + A_{21}\hat{x}_1(t)$$

Since $\hat{x}_1(t)$ is know, we can consider it a forcing term in the differential equation and this new system has eigenvalues $\mathrm{spec}(\frac{A_{22}}{\varepsilon})$. So the remaining $m$ eigenvalues of the original system tend to these as $\varepsilon \to 0$. □

b. *The system*

$$\begin{aligned} \dot{x}_1 =& \quad A_{11}x_1 + A_{12}x_2 \\ 0 =& \quad A_{21}x_1 + A_{22}x_2 \end{aligned}$$

*is referred to as the* singularly perturbed *or* low frequency approximation.

*In electronic circuits, we also have in addition to parasitic (small) capacitances, coupling (large) capacitances. These are modeled by*

$$\begin{aligned} \dot{x}_1 =& \quad A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \\ \epsilon\dot{x}_2 =& \quad A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \\ \mu\dot{x}_3 =& \quad A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \end{aligned}$$

*with $\epsilon > 0$ small and $\mu > 0$ large. A* mid frequency model *takes $\epsilon = 0, \mu = \infty$, a* low frequency model *takes $\epsilon = 0$ and sets $\mu = \infty$ in the $\tau = \frac{t}{\mu}$ time scale and a* high frequency model *sets $\mu = \infty$ and then sets $\epsilon = 0$ in the time scale $\tau = \frac{t}{\epsilon}$. Find the relationship between the eigenvalues of each if these models.*

**soln.** Now, we consider the system with three blocks, with both small and large capacitances. Let $x_i$ have dimension $n_i$.

- **Mid-freq model**: Dividing the third equation by $\mu$ and setting $\varepsilon = 0$ and letting $\mu \to \infty$ the system becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & A_{13} - A_{12}A_{22}^{-1}A_{23} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

  which has $n_1$ eigenvalues at $\mathrm{spec}(A_{11} - A_{12}A_{22}^{-1}A_{21})$ and $n_3$ eigenvalues at 0. Similar to the above, it has $n_2$ eigenvalues which tend to $\mathrm{spec}\left(\frac{A_{22}}{\varepsilon}\right)$ as $\varepsilon \to 0$.
- **Low-freq model**: Use the change of variables $t = \mu\tau$ to get

$$\frac{dx_i}{d\tau} = \frac{dx_i}{dt}\frac{dt}{d\tau} = \mu\dot{x}_i.$$

  Substituting this into the model with $\varepsilon = 0$ and $\mu \to \infty$, we obtain the following system

$$\begin{aligned} 0 &= A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \\ 0 &= A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \\ \frac{dx_3}{d\tau} &= A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \end{aligned}$$

Since the first two equations are algebraic, we can solve the first two equations for $x_1$ and $x_2$ in terms of $x_3$. These expressions can be plugged into the third equation to obtain the differential equation

$$\frac{dx_3}{d\tau} = Mx_3$$

so that the system has $n_3$ eigenvalues spec$(M)$. Thus, $n_3$ eigenvalues of the original system tend to spec$(M)$.

- **High-freq model**: Now, we use the change of variables $t = \varepsilon\tau$ and let $\mu \to \infty$ and $\varepsilon = 0$ to obtain the system

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \\ \dot{x}_3 &= 0 \end{aligned}$$

Hence the trajectories of $x_1$ and $x_3$ are approximately constant, so as $\varepsilon \to 0$, $n_2$ eigenvalues tend to those of $A_{22}$.

**Nummerical Simulation Facts for Stiff Differential Equations.**

- A problem is stiff if the stepsize is dictated by stability requirements rather than by accuracy requirements.
- A problem is stiff if explicit methods don't work, or work only extremely slowly.
- A linear problem is stiff if all of its eigenvalues have negative real part, and the stiffness ratio (the ratio of the magnitudes of the real parts of the largest and smallest eigenvalues) is large.

**Problem 3.** (Linearization of Nonlinear Systems.) Consider the nonlinear systems given below. Find the equilibrium points and determine the type of stability they exhibit.

a. $x \in \mathbb{R}$: $\dot{x} = ax - x^3$ for arbitrary $a \in \mathbb{R}$
**Solution.** The fixed points (critical points) are determined by solving

$$f(x) = x(a - x^2) = 0$$

Hence we have

$$\mathcal{C} = \{x = 0,\ x = \pm\sqrt{a}\}$$

The linear system is $\dot{x} = A(x^*)x$ where

$$A(x^*) := a - 3(x^*)^2,$$

Hence, we have

- $x^* = 0$ :

$$\dot{x} = ax$$

The eigenvalue of the Jacobian $Df|_{x=x^*}$ is $a$, so that the origin is a (locally) stable fixed point when $a < 0$ and when $a > 0$ it is unstable. For $a = 0$, we cannot draw any conclusions.

- $x^* = \pm\sqrt{a}$:

$$\dot{x} = -2ax$$

The eigenvalue of the Jacobian $Df|_{x=x^*}$ is $-2a$, so that $x^* = \pm\sqrt{a}$ are both stable if $a > 0$, and unstable if $a < 0$.

b. Let $a$ be some arbitrary parameter for the system

$$\begin{aligned} \dot{x} &= x^2 + y \\ \dot{y} &= x - y + a \end{aligned}$$

where $(x, y) \in \mathbb{R} \times \mathbb{R}$. **Solution.** Setting $\dot{x} = 0$ and $\dot{y} = 0$, we have

$$x^2 = -y,\ x = y - a \iff (y - a)^2 = -y$$

so that if $a < \frac{1}{4}$ we have two equilibrium points

$$\mathcal{C} = \left\{ \left( \frac{1}{2}(-1 - \sqrt{1-4a}), \frac{1}{2}(-1 - \sqrt{1-4a} + 2a) \right), \left( \frac{1}{2}(-1 + \sqrt{1-4a}), \frac{1}{2}(-1 + \sqrt{1-4a} + 2a) \right) \right\},$$

and if $a = \frac{1}{4}$ we have one equilibrium point at $(-\frac{1}{2}, -\frac{1}{4})$, and no equilibrium when $a > \frac{1}{4}$.
The Jacobian is

$$Df = \begin{bmatrix} 2x & 1 \\ 1 & -1 \end{bmatrix}$$

so that

$$Df|_{x^*} = \begin{bmatrix} -1 \pm \sqrt{1-4a} & 1 \\ 1 & -1 \end{bmatrix}$$

At $x = \frac{1}{2}(-1 + \sqrt{1-4a})$, the eigenvalues of the Jacobian are

$$\lambda_1, \lambda_2 = \left( \frac{1}{2}(-2 + \sqrt{1-4a} - \sqrt{5-4a}), \frac{1}{2}(-2 + \sqrt{1-4a} + \sqrt{5-4a}) \right)$$

so that this critical point is what is known as a saddle—i.e., in the range $a \in (-\infty, 1/4)$, we have one positive and one negative eigenvalue.
At $x = \frac{1}{2}(-1 - \sqrt{1-4a})$, the eigenvalues of the Jacobian are

$$\lambda_1, \lambda_2 = \left( \frac{1}{2}(-2 - \sqrt{1-4a} - \sqrt{5-4a}), \frac{1}{2}(-2 - \sqrt{1-4a} + \sqrt{5-4a}) \right) < 0$$

so that the critical point is stable since these eigenvalues are negative.
For the case when $a = \frac{1}{4}$, we have the single critical point at $(x, y) = (-\frac{1}{2}, -\frac{1}{4})$ and the Jacobian is

$$Df|_{x^*} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

so that this point is a saddle since the eigenvalues are $(-2, 0)$. We cannot apply Hartman-Grobman to this case.

**Observation.** If we think about the behavior of this system as we vary $a$ from $-\infty$ to $\frac{1}{4}$, we see that a so-called bifurcation occurs since we go from two critical points (one stable and one a saddle) to a single critical point (only a saddle).