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Topic: Solving ODEs

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1 Review

Recall

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \quad \text{(state DE)} \\ y(t) &= C(t)x(t) + D(t)u(t) \quad \text{(read-out eqn.)} \end{aligned}$$

with initial data (t_0, x_0) and the assumptions on $A(\cdot), B(\cdot), C(\cdot), D(\cdot), u(\cdot)$ all being PC:

- $A(t) \in \mathbb{R}^{n\times n}$
- $B(t) \in \mathbb{R}^{n\times m}$
- $C(t) \in \mathbb{R}^{p\times n}$
- $D(t) \in \mathbb{R}^{p\times m}$

The input function $u(\cdot) \in \mathcal{U}$, where $\mathcal{U}$ is the set of piecewise continuous functions from $\mathbb{R}_+ \to \mathbb{R}^m$.

This system satisfies the assumptions of our existence and uniqueness theorem. Indeed,

1. For all fixed $x \in \mathbb{R}^n$, the function $t \in \mathbb{R}_+ \backslash \mathcal{D} \to f(x,t) \in \mathbb{R}^n$ is continuous where $\mathcal{D}$ contains all the points of discontinuity of $A(\cdot), B(\cdot)$, $C(\cdot)$, $D(\cdot)$, $u(\cdot)$
2. There is a PC function $k(\cdot) = \|A(\cdot)\|$ such that

$$\|f(\xi,t) - f(\xi',t)\| = \|A(t)(\xi - \xi')\| \le k(t)\|\xi - \xi'\| \quad \forall t \in \mathbb{R}_+,\ \forall \xi, \xi' \in \mathbb{R}^n$$

Hence, by the above theorem, the differential equation has a unique continuous solution $x : \mathbb{R}_+ \to \mathbb{R}^n$ which is clearly defined by the parameters $(t_0, x_0, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times U$.

Theorem 1. (Existence of the state transition map/flow.) Under the assumptions and notation above, for every triple $(t_0, x_0, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times U$, the state transition map

$$x(\cdot) = \phi(\cdot, t_0, x_0, u) : \mathbb{R}_+ \to \mathbb{R}^n$$

is a continuous map well-defined as the unique solution of the state differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

with (t_0, x_0) such that $x(t_0) = x_0$ and $u(\cdot) \in U$.

2 Problems

Problem 1. (Existence and Uniqueness.) Let $A(t)$ and $B(t)$ be $n \times n$ and $n \times m$ matrices, respectively, whose whose elements are real (or complex) valued piecewise continuous functions on $\mathbb{R}_+$. Let $u(\cdot)$ be a piecewise continuous function from $\mathbb{R}_+$ to $\mathbb{R}^m$. Show that for any fixed $u(\cdot)$, the differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

satisfies the conditions of the Fundamental Theorem of ODEs (i.e. **A1** and **A2**).

Solution.

Problem 2. (Showing Uniqueness via Bellman-Gronwall.) To prove the uniqueness claim in the Fundamental Theorem of ODEs, we use the so called Bellman-Gronwall Lemma.

Lemma 2 (Bellman-Gronwall)**.** Let $u(\cdot)$, $k(\cdot)$ be real-valued, piecewise continuous functions on $\mathbb{R}_+$ and assume $u(\cdot), k(\cdot) > 0$ on $\mathbb{R}_+$. Suppose $c_1 > 0$, $t_0 \in \mathbb{R}_+$. If

$$u(t) \leq c_1 + \int_{t_0}^{t} k(\tau)u(\tau)\ d\tau$$

then

$$u(t) \leq c_1 \exp\left(\int_{t_0}^{t} k(\tau)\ d\tau\right)$$

Using the Bellman-Gronwall Lemma, show that the solution to the linear time varying differential equation given below is unique:

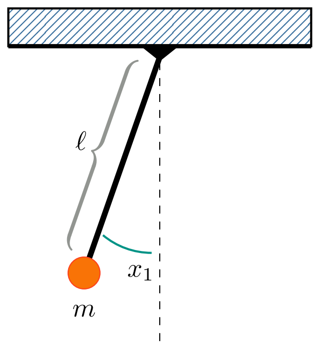
$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(t_0) &= x_0 \end{aligned}$$

Solution.

Problem 3.(Existence and Uniqueness of Solutions to Nonlinear Equations). Consider the pendulum equation with friction and constant input torque:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 + \frac{T}{m\ell^2} \end{aligned}$$

where x_1 is the angle that the pendulum makes with the vertical, x_2 is the angular rate of change, m is the mass of the bob, ℓ is the length of the pendulum, k is the friction coefficient, and T is a constant torque. Let $B_r(0) = \{x \in \mathbb{R}^2 \mid \|x\| < r\}$. For this system (represented as $\dot{x} = f(x)$) determine whether f is locally Lipschitz in x on $B_r(0)$ for sufficiently small r, locally Lipschitz in x on $B_r(0)$ for any finite r, or globally Lipschitz in x (i.e. Lipschitz for all $x \in \mathbb{R}^2$).



Solution.

Problem 4. (Floquet's Theorem). First, consider the following fact.

Fact 3. If $X(\cdot)$ and $Y(\cdot)$ are fundamental matrices of

$$\dot{x} = A(t)x$$

then there exits a constant, nonsingular matrix C such that $X(t) = Y(t)C$. Fundamental matrices are solutions to the matrix differential equation $\dot{X}(t) = A(t)X(t)$ such that $\det(X(t)) \neq 0$ for all $t \in \mathbb{R}_+$.

Consider the differential equation

$$\dot{x}(t) = A(t)x(t) \tag{1}$$

where $A(T+t) = A(t)$. Let $\Phi(t, t_0)$ be the state transition matrix. It is easy to verify (by direct substitution) that $t \mapsto \Phi(t+T, t_0)$ is also a fundamental matrix. Indeed, we have

$$\frac{d}{dt}\Phi(t+T, t_0) = A(t+T)\Phi(t+T, t_0) = A(t)\Phi(t+T, t_0).$$

The above fact implies then that

$$\Phi(t+T, t_0) = \Phi(t, t_0)C$$

so that plugging in $t = 0$ and $t_0 = 0$, we have that

$$\Phi(T, 0) = C.$$

This is the first part of Floquet's theorem! d Prove the following statements:

a. There exists a nonsingular, continuously differentiable matrix $P(t)$ with period T and a constant possibly complex matrix B such that
$$\Phi(t, t_0) = P(t)e^{B(t-t_0)}P(t_0)^{-1}$$

b. By changing to a periodically varying system of coordinates
$$x(t) = P(t)\xi(t), \quad \forall t \in \mathbb{R}_+,$$
the differenatial equation is equivalent to
$$\dot{\xi}(t) = B\xi(t) \quad \forall t \in \mathbb{R}_+.$$

You may use the following lemma.

Lemma 4 (Existence of Matrix Logarithm)**.** Let $M \in \mathbb{C}^{n\times n}$ be a square matrix. There exists a real matrix S such that $e^S = M$ if and only if M is nonsingular and for every negative eigenvalue λ of M and for every positive integer k the Jordan form of M has an even number of $k \times k$ blocks associated with λ.

More detail on this lemma can be found in **[C&D]**.

Solution.

a. *There exists a nonsingular, continuously differentiable matrix* $P(t) := \Phi(t,0)e^{-Bt}$ *with period* T *and a constant possibly complex matrix* B *such that*

$$\Phi(t, t_0) = P(t)e^{B(t-t_0)}P(t_0)^{-1}$$

b. *By changing to a periodically varying system of coordinates*

$$x(t) = P(t)\xi(t), \quad \forall t \in \mathbb{R}_+,$$

the differenatial equation is equivalent to

$$\dot{\xi}(t) = B\xi(t) \quad \forall t \in \mathbb{R}_+$$

Note: There is a discrete time counterpart to this stating that a periodically varying difference equation can be reduced to a time invariant linear difference equation via a simple change of coordinates (cf. **[C&D]**). We will also see connections to stability of nonlinear dynamical systems around a periodic orbit.