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Topic: Solving ODEs

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1 Review

Recall

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) \quad \text{(state DE)} \\ y(t) &= C(t)x(t) + D(t)u(t) \quad \text{(read-out eqn.)}\end{aligned}$$

with initial data (t_0, x_0) and the assumptions on $A(\cdot), B(\cdot), C(\cdot), D(\cdot), u(\cdot)$ all being PC:

- $A(t) \in \mathbb{R}^{n\times n}$
- $B(t) \in \mathbb{R}^{n\times m}$
- $C(t) \in \mathbb{R}^{p\times n}$
- $D(t) \in \mathbb{R}^{p\times m}$

The input function $u(\cdot) \in \mathcal{U}$, where $\mathcal{U}$ is the set of piecewise continuous functions from $\mathbb{R}_+ \to \mathbb{R}^m$.

This system satisfies the assumptions of our existence and uniqueness theorem. Indeed,

1. For all fixed $x \in \mathbb{R}^n$, the function $t \in \mathbb{R}_+ \backslash \mathcal{D} \to f(x,t) \in \mathbb{R}^n$ is continuous where $\mathcal{D}$ contains all the points of discontinuity of $A(\cdot), B(\cdot), C(\cdot), D(\cdot), u(\cdot)$
2. There is a PC function $k(\cdot) = \|A(\cdot)\|$ such that

$$\|f(\xi,t) - f(\xi',t)\| = \|A(t)(\xi - \xi')\| \le k(t)\|\xi - \xi'\| \quad \forall t \in \mathbb{R}_+, \ \forall \xi, \xi' \in \mathbb{R}^n$$

Hence, by the above theorem, the differential equation has a unique continuous solution $x : \mathbb{R}_+ \to \mathbb{R}^n$ which is clearly defined by the parameters $(t_0, x_0, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times U$.

Theorem 1. (Existence of the state transition map/flow.) Under the assumptions and notation above, for every triple $(t_0, x_0, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times U$, the state transition map

$$x(\cdot) = \phi(\cdot, t_0, x_0, u) : \mathbb{R}_+ \to \mathbb{R}^n$$

is a continuous map well-defined as the unique solution of the state differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

with (t_0, x_0) such that $x(t_0) = x_0$ and $u(\cdot) \in U$.

2 Problems

Problem 1. (Existence and Uniqueness.) Let $A(t)$ and $B(t)$ be $n \times n$ and $n \times m$ matrices, respectively, whose whose elements are real (or complex) valued piecewise continuous functions on $\mathbb{R}_+$. Let $u(\cdot)$ be a piecewise continuous function from $\mathbb{R}_+$ to $\mathbb{R}^m$. Show that for any fixed $u(\cdot)$, the differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

satisfies the conditions of the Fundamental Theorem of ODEs (i.e. **A1** and **A2**).

Solution. **solution**. Let $A(t)$ and $B(t)$ be respectively $n \times n$ and $n \times n_i$ matrices whose elements are real (or complex) valued P.C. functions on $\mathbb{R}_+$. Let $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n_i}$ be a P.C. function. Now, we must show that for fixed $u(\cdot)$, $f(x,t) = A(t)x(t) + B(t)u(t)$ is P.C. in t and L.C. in x. Since $A(t)$, $B(t)$, and $u(t)$ are P.C. in t, it is clear that $f(x,t) = A(t)x(t) + B(t)u(t)$ is P.C. in t. Note the function norm is a continuous function since given $\varepsilon > 0$, for $\delta = \frac{\varepsilon}{2}$,

$$(\|x - y\| < \delta) \Rightarrow \left(\|g(x) - g(y)\| = \|\|x\| - \|y\|\| \le \|x - y\| < \delta = \frac{\varepsilon}{2} < \varepsilon\right).$$

Now,

$$\begin{aligned}
\|f(x,t) - f(y,t)\| &= \|A(t)x(t) + B(t)u(t) - A(t)y(t) - B(t)u(t)\| \\
&= \|A(t)(x(t) - y(t))\| \\
&\le \|A(t)\|\|x(t) - y(t)\|
\end{aligned}$$

Let $k(t) = \|A(t)\|$. Since $A(t)$ is P.C. in t and $\|\cdot\|$ is continuous, $k(t)$ is P.C. in t. Thus, $f(x,t)$ is L.C. in x since $\|f(x,t) - f(y,t)\| \le \|A(t)\|\|x(t) - y(t)\|$.

Problem 2. (Showing Uniqueness via Bellman-Gronwall.) To prove the uniqueness claim in the Fundamental Theorem of ODEs, we use the so called Bellman-Gronwall Lemma.

Lemma 2 (Bellman-Gronwall)**.** Let $u(\cdot)$, $k(\cdot)$ be real-valued, piecewise continuous functions on $\mathbb{R}_+$ and assume $u(\cdot), k(\cdot) > 0$ on $\mathbb{R}_+$. Suppose $c_1 \ge 0$, $t_0 \in \mathbb{R}_+$. If

$$u(t) \le c_1 + \int_{t_0}^{t} k(\tau)u(\tau)\ d\tau$$

then

$$u(t) \le c_1 \exp\left(\int_{t_0}^{t} k(\tau)\ d\tau\right)$$

Using the Bellman-Gronwall Lemma, show that the solution to the linear time varying differential equation given below is unique:

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
x(t_0) &= x_0
\end{aligned}$$

Solution. Assume $\phi(t), \psi(t)$ are two solutions so that $\phi(t_0) = \psi(t_0) = x_0$ and

$$\begin{aligned}
\dot{\phi}(t) &= A(t)\phi(t) + B(t)u(t) \\
\dot{\psi}(t) &= A(t)\psi(t) + B(t)u(t)
\end{aligned}$$

Then

$$\phi(t) - \psi(t) = \int_{t_0}^{t} (A(\tau)\phi(\tau) - A(\tau)\psi(\tau))\ d\tau$$

so that

$$\|\phi(t) - \psi(t)\| \le \|A(t)\|_{\infty,[t_0,t]} \int_{t_0}^{t} \|\phi(\tau) - \psi(\tau)\| \, d\tau$$

By Bellman-Gronwall,

$$\|\phi(t) - \psi(t)\| \le c_1 + \|A(t)\|_{\infty,[t_0,t]} \int_{t_0}^{t} \|\phi(\tau) - \psi(\tau)\| \, d\tau$$

implies

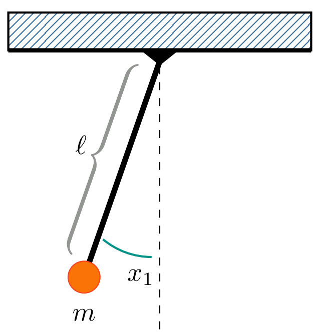
$$\|\phi(t) - \psi(t)\| \le c_1 \exp\left(\|A(t)\|_{\infty,[t_0,t]}(t - t_0)\right)$$

This is true for all $c_1 \ge 0$, so set $c_1 = 0 \ldots$

Problem 3.(Existence and Uniqueness of Solutions to Nonlinear Equations). Consider the pendulum equation with friction and constant input torque:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 + \frac{T}{m\ell^2} \end{aligned}$$

where x_1 is the angle that the pendulum makes with the vertical, x_2 is the angular rate of change, m is the mass of the bob, ℓ is the length of the pendulum, k is the friction coefficient, and T is a constant torque. Let $B_r(0) = \{x \in \mathbb{R}^2 \mid \|x\| < r\}$. For this system (represented as $\dot{x} = f(x)$) determine whether f is locally Lipschitz in x on $B_r(0)$ for sufficiently small r, locally Lipschitz in x on $B_r(0)$ for any finite r, or globally Lipschitz in x (i.e. Lipschitz for all $x \in \mathbb{R}^2$).



Solution. solution. Consider the D.E.

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -\tfrac{g}{\ell}\sin x_1 - \tfrac{k}{m}x_2 + \tfrac{T}{m\ell^2} \end{aligned} \tag{1}$$

Claim: f is Globally Lipschitz.
Since finite dimensional norms are equivalent, it is sufficient to consider the matrix norm $\|\cdot\|_1$, i.e. the max-column sum norm.

$$\|\partial_x f\|_1 = \left\| \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{pmatrix} \right\|_1 = \max\left\{ \left|-\frac{g}{l}\cos x_1\right|, |1| + \left|-\frac{k}{m}\right| \right\} \le \max\left\{ \left|-\frac{g}{l}\right|, |1| + \left|-\frac{k}{m}\right| \right\}$$

By the mean value theorem we have

$$\|f(x,t) - f(y,t)\| \le \|\partial_x f\| \|x - y\|$$

so that $f(x,t) : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^2$ is globally Lipschitz in x for all t since we have a P.C. function $k(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ where $k(t) = \max\left\{ \left|-\frac{g}{l}\right|, |1| + \left|-\frac{k}{m}\right| \right\}$ such that

$$\|f(x,t) - f(y,t)\| \le \|\partial_x f\| \|x - y\| \le \max\left\{ \left|-\frac{g}{l}\right|, |1| + \left|-\frac{k}{m}\right| \right\} \|x - y\|$$

So, since f is globally L.C., it is locally L.C. on $B_r = \{x \in \mathbb{R}^2 : \|x\| < r\}$ for both sufficiently small r and $r < \infty$.

Problem 4. (Floquet's Theorem). First, consider the following fact.

Fact 3. If $X(\cdot)$ and $Y(\cdot)$ are fundamental matrices of

$$\dot{x} = A(t)x$$

then there exits a constant, nonsingular matrix C such that $X(t) = Y(t)C$. Fundamental matrices are solutions to the matrix differential equation $\dot{X}(t) = A(t)X(t)$ such that $\det(X(t)) \neq 0$ for all $t \in \mathbb{R}_+$.

Consider the differential equation

$$\dot{x}(t) = A(t)x(t) \tag{2}$$

where $A(T+t) = A(t)$. Let $\Phi(t, t_0)$ be the state transition matrix. It is easy to verify (by direct substitution) that $t \mapsto \Phi(t+T, t_0)$ is also a fundamental matrix. Indeed, we have

$$\frac{d}{dt}\Phi(t+T, t_0) = A(t+T)\Phi(t+T, t_0) = A(t)\Phi(t+T, t_0).$$

The above fact implies then that

$$\Phi(t+T, t_0) = \Phi(t, t_0)C$$

so that plugging in $t = 0$ and $t_0 = 0$, we have that

$$\Phi(T, 0) = C.$$

This is the first part of Floquet's theorem! d Prove the following statements:

a. There exists a nonsingular, continuously differentiable matrix $P(t)$ with period T and a constant possibly complex matrix B such that

$$\Phi(t, t_0) = P(t)e^{B(t-t_0)}P(t_0)^{-1}$$

b. By changing to a periodically varying system of coordinates

$$x(t) = P(t)\xi(t), \quad \forall t \in \mathbb{R}_+,$$

the differenatial equation is equivalent to

$$\dot{\xi}(t) = B\xi(t) \quad \forall t \in \mathbb{R}_+.$$

You may use the following lemma.

Lemma 4 (Existence of Matrix Logarithm)**.** Let $M \in \mathbb{C}^{n\times n}$ be a square matrix. There exists a real matrix S such that $e^S = M$ if and only if M is nonsingular and for every negative eigenvalue λ of M and for every positive integer k the Jordan form of M has an even number of $k \times k$ blocks associated with λ.

More detail on this lemma can be found in **[C&D]**.

Solution.

a. *There exists a nonsingular, continuously differentiable matrix $P(t) := \Phi(t,0)e^{-Bt}$ with period T and a constant possibly complex matrix B such that*

$$\Phi(t,t_0) = P(t)e^{B(t-t_0)}P(t_0)^{-1}$$

By Lemma 4, there exists a matrix $B := \frac{1}{T}\log(\Phi(T,0)) \in \mathbb{C}^{n\times n}$ such that

$$C = e^{BT}$$

so that $\Phi(T) = C = e^{BT}$. Define $P(t) := \Phi(t)e^{-Bt}$. We can do this since $\Phi(t)$ is a well defined solution and e^{-Bt} exists by Lemma 4. Hence, we have that

$$P(t+T) = \Phi(T+t,0)e^{-B(t+T)} = \Phi(t,0))Ce^{-BT}e^{-Bt} = \Phi(t,0)e^{-Bt} = P(t),$$

which shows that our candidate for $P(t)$ is indeed periodic with period T.

Now, we have that

$$\begin{aligned}\Phi(t,t_0) &= \Phi(t,0)\Phi(0,t_0)\\ &= \Phi(t,0)(\Phi(t_0,0))^{-1} \quad \text{(since } \Phi(t,\tau)^{-1} = \Phi(\tau,t)\text{, try and prove this)}\\ &= P(t)e^{Bt}e^{-Bt_0}P(t_0)^{-1}\\ &= P(t)e^{B(t-t_0)}P(t_0)^{-1}.\end{aligned}$$

b. *By changing to a periodically varying system of coordinates*

$$x(t) = P(t)\xi(t), \quad \forall t \in \mathbb{R}_+,$$

the differenatial equation is equivalent to

$$\dot{\xi}(t) = B\xi(t) \quad \forall t \in \mathbb{R}_+$$

If $x(t)$ is any solution to $\dot{x} = A(t)x$ then for some $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{C}^n$, we have $x(t) = \Phi(t,t_0)x_0$. Using the change of coordinates, we have that

$$\xi(t) = P(t)^{-1}x(t) = P(t)^{-1}\Phi(t,t_0)x_0 = P(t)^{-1}\Phi(t,t_0)P(t_0)\xi_0 = e^{B(t-t_0)}\xi_0,$$

where $\xi_0 := P(t_0)^{-1}x_0$. This shows that $\xi(\cdot)$ satisfies the differential equation $\dot{\xi} = B\xi$.

Note: There is a discrete time counterpart to this stating that a periodically varying difference equation can be reduced to a time invariant linear difference equation via a simple change of coordinates (cf. **[C&D]**). We will also see connections to stability of nonlinear dynamical systems around a periodic orbit.