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Topic: Solving ODEs

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1 Review

Continuous time LTI Systems. Consider now the general LTI system in state-space form:

$$\dot{x} = Ax + Bu \tag{1}$$
$$y = Cx + Du \tag{2}$$

where

- $x \in \mathbb{R}^n$ is the "state" of the system
- $u \in \mathbb{R}^m$ is the "input" to the system
- $y \in \mathbb{R}^p$ is the "output" of the system
- $A \in \mathbb{R}^{n\times n}$ describes how the state changes in time (dynamics)
- $B \in \mathbb{R}^{n\times m}$ describes how the input effects the state dynamics
- $C \in \mathbb{R}^{p\times n}$ describes how the state is transformed to the output
- $D \in \mathbb{R}^{p\times m}$ describes how the input is transformed to the output (for the most part in this class we take $D = 0$).

Discrete time LTI Systems. A discrete time LTI system is given by

$$x[k+1] = Ax[k] + Bu[k] \tag{3}$$
$$y[k] = Cx[k] + Du[k] \tag{4}$$

2 Problems

Problem 1. (Sampled Data System.) You are given a time-invariant system

$$\dot{x} = Ax + Bu$$

that is sampled every T seconds. Denote $x(kT)$ by x_k. Further, the input u is held constant between kT and $(k+1)T$, that is, $u(t) = u_k$ for $t \in [kT, (k+1)T]$. Derive the state equation for the sampled data system, that is, give a formula for x_{k+1} in terms of x_k and u_k.

Solution.

Problem 2. Matrix Differential Equations. Consider the matrix differential equation

$$\dot{X} = A_1 X + X A_2^*, \quad X(t_0) = X_0$$

Show the solution is

$$X(t) = e^{A_1(t-t_0)} X_0 \left(e^{A_2(t-t_0)}\right)^* \tag{5}$$

Solution.

Problem 3. Properties of State Transition Matrices. We will see in the next recorded lecture that for a linear time varying system

$$\dot{x} = A(t)x, \quad x(t_0) = x_0$$

that the solution is given by

$$x(t) = \Phi(t, t_0)x_0$$

where $\Phi(t, t_0)$ is what is known as the state transition matrix. It is a generalization of the matrix exponential to the time varying case.

With this in mind, consider the differential equation

$$\dot{x} = (A + B)x$$

Show that the state transition matrix is

$$e^{At}\Phi_M(t, t_0)e^{-At_0} \tag{6}$$

where

$$M(t) = e^{-At}Be^{At},$$

and $\Phi_M(t, t_0)$ is the state transition matix of the differential equation

$$\dot{z} = M(t)z$$

Solution.

Problem 4. Dyadic Expansions One way to understand the effects of different modes of the input (i.e., poles of the transfer function) on the state and output of the system is to express the solution in terms of a dyadic expansion. Taking the Laplace transform of our dynamical system we have

$$sX = AX + BU \iff X = (sI - A)^{-1}BU \implies Y = (C(sI-A)^{-1}B + D)U \iff H = \frac{Y}{U} = C(sI-A)^{-1}B + D$$

Suppose that A is semi-simple[1] and has dyadic expansion

$$A = E\Lambda N^* = \sum_{i=1}^{n} \lambda_i e_i \eta_i^*$$

where

$$E = \begin{bmatrix} | & \cdots & | \\ e_1 & \cdots & e_n \\ | & \cdots & | \end{bmatrix}, \quad N^* = \begin{bmatrix} -- & \eta_1^* & -- \\ \vdots & \vdots & \vdots \\ -- & \eta_n^* & -- \end{bmatrix}$$

are nonsingular matrices such that $EN^* = N^*E = I$ and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

a. Find an expression for $H(s)$ in terms of the dyadic expansion.
b. Show that
$$x(t) = \sum_{i=1}^{n} e_i \exp(\lambda_i t) \left(\langle \eta_i, x_0 \rangle + \int_0^t e^{-\lambda_i \tau} \langle B^* \eta_i, u(\tau) \rangle \, d\tau \right)$$
Find an expression for the output $y(t)$ given that
$$y = Cx + Du$$
c. Now, suppose that for some $p \in \mathbb{C}^m$ and $x_0 = 0$, that $u(t) = p\delta(t)$. That is for $k \in [m]$, the k-th scalar input is an impulse of area p_k applied at $t = 0$. Find an expression for the state.

Solution. **a**. *Find an expression for $H(s)$ in terms of the dyadic expansion.*

[1] A is semisimple if it has an eigenbasis, i.e., the geometric multiplicity of each eigenvalue of L equals its algebraic multiplicity.

b. *Show that*

$$x(t) = \sum_{i=1}^{n} e_i \exp(\lambda_i t) \left(\langle \eta_i, x_0 \rangle + \int_0^t e^{-\lambda_i \tau} \langle B^* \eta_i, u(\tau) \rangle \, d\tau \right)$$

Find an expression for the output $y(t)$ given that

$$y = Cx + Du$$

c. *Now, suppose that for some $p \in \mathbb{C}^m$ and $x_0 = 0$, that $u(t) = p\delta(t)$. That is for $k \in [m]$, the k-th scalar input is an impulse of area p_k applied at $t = 0$. Find an expression for the state.*

Some observations:

- $\langle B^* \eta_i, p \rangle$ measures the coupling between the impulsive vector input $p\delta(t)$ and the i th-mode; in particular, if $\langle B^* \eta_i, p \rangle = 0$ then the i-th mode is not excited by that particular input.
- If $B^* \eta_i = 0$, then by the expression for $x(t)$ we see that no input can excite the i-th mode, i.e. the actuators are not coupled to the i-th mode.
- if $Ce_i = 0$, then the expression for y shows that the i–th mode does not contribute to the output, i.e. the sensors are not coupled to the i-th mode.