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Topic: Solving ODEs

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1 Review

Continuous time LTI Systems. Consider now the general LTI system in state-space form:

$$\dot{x} = Ax + Bu \tag{1}$$
$$y = Cx + Du \tag{2}$$

where

- $x \in \mathbb{R}^n$ is the "state" of the system
- $u \in \mathbb{R}^m$ is the "input" to the system
- $y \in \mathbb{R}^p$ is the "output" of the system
- $A \in \mathbb{R}^{n\times n}$ describes how the state changes in time (dynamics)
- $B \in \mathbb{R}^{n\times m}$ describes how the input effects the state dynamics
- $C \in \mathbb{R}^{p\times n}$ describes how the state is transformed to the output
- $D \in \mathbb{R}^{p\times m}$ describes how the input is transformed to the output (for the most part in this class we take $D = 0$).

The solution to the CT LTI system in (1) is

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\ d\tau$$

Discrete time LTI Systems. A discrete time LTI system is given by

$$x[k+1] = Ax[k] + Bu[k] \tag{3}$$
$$y[k] = Cx[k] + Du[k] \tag{4}$$

The solution for the DT LTI system is given by

$$x[k] = A^{k-k_0}x[k_0] + \sum_{\ell=k_0}^{k-1} A^{k-\ell-1}Bu[\ell]$$

2 Problems

Problem 1. (Sampled Data System.) You are given a time-invariant system

$$\dot{x} = Ax + Bu$$

that is sampled every T seconds. Denote $x(kT)$ by x_k. Further, the input u is held constant between kT and $(k+1)T$, that is, $u(t) = u_k$ for $t \in [kT, (k+1)T]$. Derive the state equation for the sampled data system, that is, give a formula for x_{k+1} in terms of x_k and u_k.

Solution. First, for the LTI continuous time system $\Phi(t, 0) = \exp(At)$. Since $x_k = x(kT)$,

$$\begin{aligned} x_{k+1} = x((k+1)T) &= \underbrace{\exp(A((k+1)T - kT))}_{\exp(AT)} x_k + \int_{kT}^{(k+1)T} \exp(A((k+1)T - \tau))Bu_k \, d\tau \\ &= \exp(AT)x_k + \int_0^T \exp(At') \, dt' Bu_k \end{aligned}$$

So, the state equation for the sampled data system is

$$x_{k+1} = \tilde{A}x_k + \tilde{B}u_k$$

where

$$\tilde{A} = \exp(AT), \quad \tilde{B} = \int_0^T \exp(At') \, dt'$$

Problem 2. Matrix Differential Equations. Consider the matrix differential equation

$$\dot{X} = A_1 X + X A_2^*, \quad X(t_0) = X_0$$

Show the solution is

$$X(t) = e^{A_1(t-t_0)} X_0 \left(e^{A_2(t-t_0)}\right)^* \tag{5}$$

Solution. One the biggest "tricks" or proof methods we will use in this class to show two things are equivalent is the uniqueness aspect of the fundamental theorem of ODEs (M1-RL2). If two functions say $f(t)$ and $g(t)$ solve the same ODE and satisfy the same initial condition, then we can invoke the fundamental theorem of ODEs to conclude they are equivalent.

We will use this for this problem. We need to check the right and left hand side of (5) solve the same ODE and have the same initial condition. Trivially $X(t)$ solves the ODE and $X(t_0) = X_0$ (this is just by definition). Hence, we can check that the right hand side $e^{A_1(t-t_0)} X_0 e^{A_2^*(t-t_0)}$ solves the ODE and has the same initial condition. The initial condition is trivial:

$$e^{A_1(t_0-t_0)} X_0 \left(e^{A_2(t_0-t_0)}\right)^* = I \cdot X_0 \cdot I = X_0$$

Now, for the ODE:

$$\frac{d}{dt}\left(e^{A_1(t-t_0)} X_0 \left(e^{A_2(t-t_0)}\right)^*\right) = A_1 e^{A_1(t-t_0)} X_0 \left(e^{A_2(t-t_0)}\right)^* + e^{A_1(t-t_0)} X_0 \left(e^{A_2(t-t_0)}\right)^* A_2^*$$

where the second term follows from the fact that

$$\frac{d}{dt}\left(e^{A_2(t-t_0)}\right)^* = \left(\frac{d}{dt} e^{A_2(t-t_0)}\right)^* = \left(A_2 e^{A_2(t-t_0)}\right)^* = \left(e^{A_2(t-t_0)}\right)^* A_2^*$$

To complete the proof, we invoke the fundamental theorem of ODEs, specifically the fact that solutions are unique. Hence, the claimed solution holds

$$X(t) = e^{A_1(t-t_0)} X_0 \left(e^{A_2(t-t_0)}\right)^*$$

Analogously, we note that

$$\left(e^{A_2(t-t_0)}\right)^* = e^{A_2^*(t-t_0)},$$

and

$$\frac{d}{dt} e^{A_2^*(t-t_0)} = A_2^* e^{A_2^*(t-t_0)}$$

but since A_2^* commutes with $e^{A_2^*(t-t_0)}$ (by Cayley Hamilton), we have that

$$A_2^* e^{A_2^*(t-t_0)} = e^{A_2^*(t-t_0)} A_2^*.$$

So we could have argued this way too.

Problem 3. Properties of State Transition Matrices. We will see in the next recorded lecture that for a linear time varying system

$$\dot{x} = A(t)x, \quad x(t_0) = x_0$$

that the solution is given by

$$x(t) = \Phi(t, t_0)x_0$$

where $\Phi(t, t_0)$ is what is known as the state transition matrix. It is a generalization of the matrix exponential to the time varying case.

With this in mind, consider the differential equation

$$\dot{x} = (A + B)x$$

Show that the state transition matrix is

$$e^{At}\Phi_M(t, t_0)e^{-At_0} \tag{6}$$

where

$$M(t) = e^{-At}Be^{At},$$

and $\Phi_M(t, t_0)$ is the state transition matix of the differential equation

$$\dot{z} = M(t)z$$

Solution. First we know that the state transition matrix will be

$$e^{(A+B)(t-t_0)} = e^{At}\Phi_M(t, t_0)e^{-At_0}$$

by checking the solve the same (matrix) ODE and initial condition. The initial condition should be I since these are fundamental matrices. Just plugging in $t = t_0$, we see that both sides evaluate to identity. Clearly,

$$\dot{X} = \frac{d}{dt}\left(e^{(A+B)(t-t_0)}\right) = (A + B)e^{(A+B)(t-t_0)} = (A + B)X$$

Now lets check that $e^{At}\Phi_M(t, t_0)e^{-At_0}$ solves that same ODE. Indeed,

$$\begin{aligned}
\frac{d}{dt}\left(e^{At}\Phi_M(t, t_0)e^{-At_0}\right) &= Ae^{At}\Phi_M(t, t_0)e^{-At_0} + e^{At}\dot{\Phi}_M(t, t_0)e^{-At_0} \\
&= Ae^{At}\Phi_M(t, t_0)e^{-At_0} + e^{At}M(t)\Phi_M(t, t_0)e^{-At_0} \\
&= Ae^{At}\Phi_M(t, t_0)e^{-At_0} + e^{At}e^{-At}Be^{At}\Phi_M(t, t_0)e^{-At_0} \\
&= (A + B)e^{At}\Phi_M(t, t_0)e^{-At_0}
\end{aligned}$$

Hence, they solve the same ODE and thus, the claimed state transition matrix holds.

Problem 4. Dyadic Expansions One way to understand the effects of different modes of the input (i.e., poles of the transfer function) on the state and output of the system is to express the solution in terms of a dyadic expansion. Taking the Laplace transform of our dynamical system we have

$$sX = AX + BU \iff X = (sI - A)^{-1}BU \implies Y = (C(sI-A)^{-1}B + D)U \iff H = \frac{Y}{U} = C(sI-A)^{-1}B + D$$

Suppose that A is semi-simple[1] and has dyadic expansion

$$A = E\Lambda N^* = \sum_{i=1}^{n} \lambda_i e_i \eta_i^*$$

where

$$E = \begin{bmatrix} | & \cdots & | \\ e_1 & \cdots & e_n \\ | & \cdots & | \end{bmatrix}, \quad N^* = \begin{bmatrix} -- & \eta_1^* & -- \\ \vdots & \vdots & \vdots \\ -- & \eta_n^* & -- \end{bmatrix}$$

are nonsingular matrices such that $EN^* = N^*E = I$ and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

a. Find an expression for $H(s)$ in terms of the dyadic expansion.

b. Show that

$$x(t) = \sum_{i=1}^{n} e_i \exp(\lambda_i t) \left(\langle \eta_i, x_0 \rangle + \int_0^t e^{-\lambda_i \tau} \langle B^* \eta_i, u(\tau) \rangle \, d\tau \right)$$

Find an expression for the output $y(t)$ given that

$$y = Cx + Du$$

c. Now, suppose that for some $p \in \mathbb{C}^m$ and $x_0 = 0$, that $u(t) = p\delta(t)$. That is for $k \in [m]$, the k-th scalar input is an impulse of area p_k applied at $t = 0$. Find an expression for the state.

Solution. a. *Find an expression for $H(s)$ in terms of the dyadic expansion.*

$$\begin{aligned}
(sI - A)^{-1} = (sI - E\Lambda N^*)^{-1} &= (sEN^* - E\Lambda N^*)^{-1} \\
&= (E(sI - \Lambda)N^*)^{-1} \\
&= (N^*)^{-1}(sI - \Lambda)^{-1}E^{-1} \\
&= \begin{bmatrix} -- & \eta_1^* & -- \\ \vdots & \vdots & \vdots \\ -- & \eta_n^* & -- \end{bmatrix}^{-1} \operatorname{diag}\left(\frac{1}{s - \lambda_1}, \ldots, \frac{1}{s - \lambda_n} \right) \begin{bmatrix} | & \cdots & | \\ e_1 & \cdots & e_n \\ | & \cdots & | \end{bmatrix}^{-1} \\
&= \begin{bmatrix} | & \cdots & | \\ e_1 & \cdots & e_n \\ | & \cdots & | \end{bmatrix} \operatorname{diag}\left(\frac{1}{s - \lambda_1}, \ldots, \frac{1}{s - \lambda_n} \right) \begin{bmatrix} -- & \eta_1^* & -- \\ \vdots & \vdots & \vdots \\ -- & \eta_n^* & -- \end{bmatrix} \\
&= \begin{bmatrix} | & \cdots & | \\ e_1 & \cdots & e_n \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} -- & \frac{1}{s-\lambda_1}\eta_1^* & -- \\ \vdots & \vdots & \vdots \\ -- & \frac{1}{s-\lambda_n}\eta_n^* & -- \end{bmatrix} \\
&= \sum_{i=1}^{n} \frac{1}{s - \lambda_i} e_i \eta_i^*
\end{aligned}$$

[1] A is semisimple if it has an eigenbasis, i.e., the geometric multiplicity of each eigenvalue of L equals its algebraic multiplicity.

since $N^*E = I = EN^*$. Hence we have that

$$H(s) = \sum_{i=1}^{n} (s - \lambda_i)^{-1} C e_i \eta_i^* B + D$$

where $Ce_i\eta_i^*B$ is a dyad with column Ce_i and row η_i^*B. Hence a pole contribution at λ_i disappears if and only if the corresponding dyad is zero or equivalently $Ce_i = 0$ or $\eta_i^*B = 0$. Also, the nonzero vectors Ce_i and η_i^*B represent the strength of the coupling of the i–th mode with the output and the input respectively. In terms of practice, Ce_i depends on the location and sensitivity of the sensors while η_i^*B depends on the location and strength of actuators.

b. *Show that*

$$x(t) = \sum_{i=1}^{n} e_i \exp(\lambda_i t) \left(\langle \eta_i, x_0 \rangle + \int_0^t e^{-\lambda_i \tau} \langle B^* \eta_i, u(\tau) \rangle \, d\tau \right)$$

Find an expression for the output $y(t)$ given that

$$y = Cx + Du$$

Let $t_0 = 0$. We know that

$$x(t) = e^{At}x_0 = e^{E\Lambda N^* t}x_0 + \int_0^t e^{E\Lambda N^* \tau} Bu(\tau) d\tau = Ee^{\Lambda t}N^*x_0 + \int_0^t Ee^{\Lambda \tau}N^*Bu(\tau)d\tau$$

Hence

$$x(t) = \sum_{i=1}^{n} e_i e^{\lambda_i t} \left(\langle \eta_i, x_0 \rangle + \int_0^t e^{-\lambda_i \tau} \langle B^* \eta_i, u(\tau) \rangle \, d\tau \right)$$

as claimed. Hence,

$$\sum_{i=1}^{n} Ce_i e^{\lambda_i t} \left(\langle \eta_i, x_0 \rangle + \int_0^t e^{-\lambda_i \tau} \langle B^* \eta_i, u(\tau) \rangle \, d\tau \right) + Du(t)$$

c. *Now, suppose that for some $p \in \mathbb{C}^m$ and $x_0 = 0$, that $u(t) = p\delta(t)$. That is for $k \in [m]$, the k-th scalar input is an impulse of area p_k applied at $t = 0$. Find an expression for the state.*

Suppose $u(t) = p\delta(t)$ and $x_0 = 0$. Then

$$\begin{aligned} x(t) &= \sum_{i=1}^{n} e_i e^{\lambda_i t} \left(\langle \eta_i, x_0 \rangle + \int_0^t e^{-\lambda_i \tau} \langle B^* \eta_i, p\delta(\tau) \rangle \, d\tau \right) \\ &= \sum_{i=1}^{n} e_i \langle B^* \eta_i, p \rangle e^{-\lambda_i t} \end{aligned}$$

since

$$\left(\langle \eta_i, x_0 \rangle + \int_0^t e^{-\lambda_i \tau} \langle B^* \eta_i, p\delta(\tau) \rangle \, d\tau \right) = \langle B^* \eta_i, p \rangle$$

Some observations:

- $\langle B^*\eta_i, p \rangle$ measures the coupling between the impulsive vector input $p\delta(t)$ and the i th-mode; in particular, if $\langle B^*\eta_i, p \rangle = 0$ then the i-th mode is not excited by that particular input.
- If $B^*\eta_i = 0$, then by the expression for $x(t)$ we see that no input can excite the i-th mode, i.e. the actuators are not coupled to the i-th mode.
- if $Ce_i = 0$, then the expression for y shows that the i–th mode does not contribute to the output, i.e. the sensors are not coupled to the i-th mode.