You are *not* allowed to work with your peers. Show all work and derivations to receive credit. Write clearly and legibly for your own benefit.

Name : ______________________________

SID : ______________________________

Problem 1 (Two-point boundary). Suppose that the boundary conditions for $\dot{x} = A(t)x(t)$ were specified in part at t_0 and in part at t_1. In particular suppose that

$$Mx(t_0) + Nx(t_1) = b$$

with Rank $(M, N) = n$ (dimension of x). Show that this *two point boundary value problem* has a unique solution if $M + N\Phi(t_1, t_0)$ is nonsingular.

Solution

Problem 2 (Lyapunov Stability). Consider the nonlinear system

$$\dot{x} = f(x), \quad f : \mathbb{R}^n \to \mathbb{R}^n$$

where $x = 0$ is an equilibrium point for the system—i.e., $f(0) = 0$. We know that by the mean value theorem $f(x) = f(0) + \frac{df}{dx}(z)x$ where z is any point on the line segment connecting x to the origin. Since $f(0) = 0$ we have that

$$f(x) = \frac{df}{dx}(z)x = \frac{df}{dx}(0)x + \left(\frac{df}{dx}(z) - \frac{df}{dx}(0)\right)x,$$

so that $f(x) = Ax + g(x)$, where

$$A := \frac{df}{dx}(0) \quad \text{and} \quad g(x) := \left(\frac{df}{dx}(z) - \frac{df}{dx}(0)\right)x$$

Since $g(x)$ satisfies

$$\|g(x)\| \leq \left\|\left(\frac{df}{dx}(z) - \frac{df}{dx}(0)\right)\right\| \|x\|,$$

by continuity of df/dx we have that $\frac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$. This is equivalent to the following: for any $\alpha > 0$ there exists an $r > 0$ such that

$$\|g(x)\| \leq \alpha\|x\| \quad \forall \; \|x\| < r.$$

Suppose that A be Hurwitz stable (all eigenvalues have negative real part). For which values of α is the origin of the nonlinear system $\dot{x} = Ax + g(x)$ is globally asymptotically stable? That is, find an explicit upper bound $\bar{\alpha}$ on α such that for $\|g(x)\| \leq \alpha\|x\|$, the origin is globally asymptotically stable.

Hint: try constructing a Lyapunov function for the system $\dot{x} = Ax + g(x)$ using the standard quadratic form for Lyapunov functions for linear systems.

Solution.

Problem 3 (Controllability and Lyapunov) . Consider the linear system

$$\dot{x} = Ax + Bu$$

where (A, B) is controllable. Let $P = \int_0^\tau e^{-At} B B^\top e^{-A^\top t}\, dt$ for some $\tau > 0$. Show that P is positive definite. Define $K := B^\top P^{-1}$ and use $V(x) = x^\top P^{-1} x$ as a Lyapunov function for the system $\dot{x} = (A - BK)x$ to show that $(A - BK)$ is Hurwitz stable. **Hint**: Check that the derivative $\frac{d}{dt}V$ is non-positive (i.e., $\dot{V} \leq 0$). Then argue that in fact it has to be strictly negative by constructing a contradiction. Note this is similar to the control synthesis we saw in the notes, however, you are asked to directly show that $V(x) = x^\top P^{-1} x$ is a Lyapunov function versus arguing that $(-\mu I - A)$ is stable and so on...

Solution.

Problem 4 (Part I. Controllability and Observability). Consider the system

$$\begin{aligned} \dot{x} &= -ax + ku \\ y &= x \end{aligned}$$

i.e. $A = -a$, $B = k$, $C = 1$ and $D = 0$. Is the system controllable? observable? Consider the system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} u \\ y &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

a. When is the system controllable? (provide proof)

b. When is the system observable? (provide proof)

c. By referring to the definitions of observability and controllability from part a, briefly explain these results.

Solution.

Problem 4 (Part II. Controllability/Reachability) . Consider the system

$$\dot{x} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

True or False: there exists a $u(\tau)$ defined on the interval $[0, t]$ such that $x(t) = [10, -10, 0]^\top$. If true provide justification (i.e., a proof) and if false, explain why.

Solution.