Mod3-RL5: Stabilizability & Detectability

References:

- Chapter 8 Callier & Desoer **[C&D]**
- Chapter 11 and 15 Hespanha **[JH]**
- **[510]** Lecture Notes (Finite Rank Operator Lemma, Hilbert Spaces, Adjoints, etc.)

Controllable/Observable Decomposition

When a system is not controllable (resp. not observable), it is often useful to be able to decompose the state space (and system) into the controllable and uncontrollable subspaces (resp., observable and unobservable subspaces). While the details can be found in the lectures notes, here we simply recall the main results:

Proposition. Both Ker($\mathscr{O}$) and Im($\mathscr{C}$) are A-invariant.

Given this proposition, we can

create a similarity transformation taking an uncontrollable system to the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{\mathrm{c}} & A_{12} \\ 0 & A_{\mathrm{uc}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{\mathrm{c}} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{\mathrm{c}} & C_{\mathrm{uc}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

create a similarity transformation taking an unobservable system to the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{\mathrm{o}} & 0 \\ A_{21} & A_{\mathrm{uo}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{\mathrm{o}} \\ B_{\mathrm{uo}} \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{\mathrm{o}} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

With these fundamental decompositions we can discuss the concepts of stabilizability and detectabiility. And, more to the point observer and controller synthesis.

Stabilizability

Given a system that is not controllable, we would like to still be able to give guarantees on when we can stabilize the system. Intuitively, we should expect that we can stabilize a system if all the uncontrollable modes (or states) are already stable. This is the concept of stabilizability.

Similarity transform to controllable decomposition:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_{\mathrm{c}} & A_{12}\\ 0 & A_{\mathrm{uc}}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}B_{\mathrm{c}}\\ 0\end{bmatrix} u$$

$$y = \begin{bmatrix}C_{\mathrm{c}} & C_{\mathrm{uc}}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

- Controllable states x_1 and uncontrollable states x_2

Definition: The pair (A, B) is stabilizable if there is a similarity transformation to the form above with A_{uc} Hurwitz stable.

Stabilizability Theorem

Theorem. The following are equivalent:

a. The pair (A, B) is stabilizable
b. Every eigenvector of $A^\top$ corresponding to an eigenvalue with positive or zero real part is not in the kernel of $B^\top$
c. (PBH Test) $\text{rank}([A - \lambda I \; B]) = n$, $\forall \lambda \in \mathbb{C}$ such that $\text{Re}(\lambda) \geq 0$.
d. There is a positive definite solution $P = P^\top \succ 0$ to the Lyapunov matrix inequality

$$AP + PA^\top - BB^\top \prec 0$$

Like with controllability we can leverage the Lyapunov test for stabilizability in d above to synthesize stabilizing feedback controllers.

Synthesizing Controllers

Consider the system $\dot{x} = Ax + Bu$ and suppose the system is stabilizable

Detectability

Given a system that is not observable, we would like to still be able to give guarantees on when we can unobservable states are well behaved.

Similarity transform to observable decomposition:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_\text{o} & 0 \\ A_{21} & A_\text{uo} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_\text{o} \\ B_\text{uo} \end{bmatrix} u$$

$$y = \begin{bmatrix} C_\text{o} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Observable states x_1 and unobservable states x_2

Definition: The pair (A, C) is detectable if there is a similarity transformation to the form above with A_uo Hurwitz stable.

Detectability Theorem

Theorem. The following are equivalent:

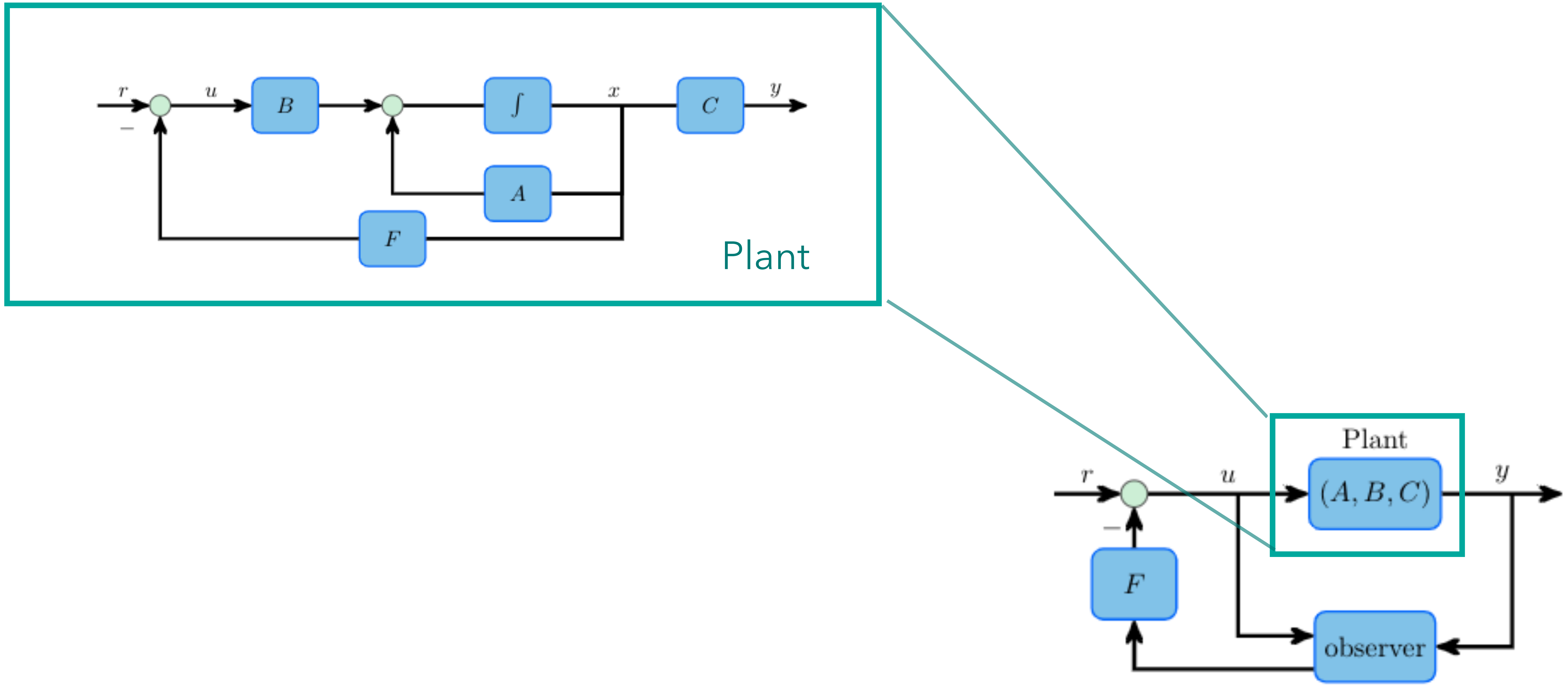
a. The pair (A, C) is detectable
b. Every eigenvector of A corresponding to an eigenvalue with positive or zero real part is not in the kernel of C
c. (PBH Test) rank $\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = n, \forall \lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) \geq 0$.
d. There is a positive definite solution $P = P^\top \succ 0$ to the Lyapunov matrix inequality

$$AP + PA^\top - C^\top C \prec 0$$

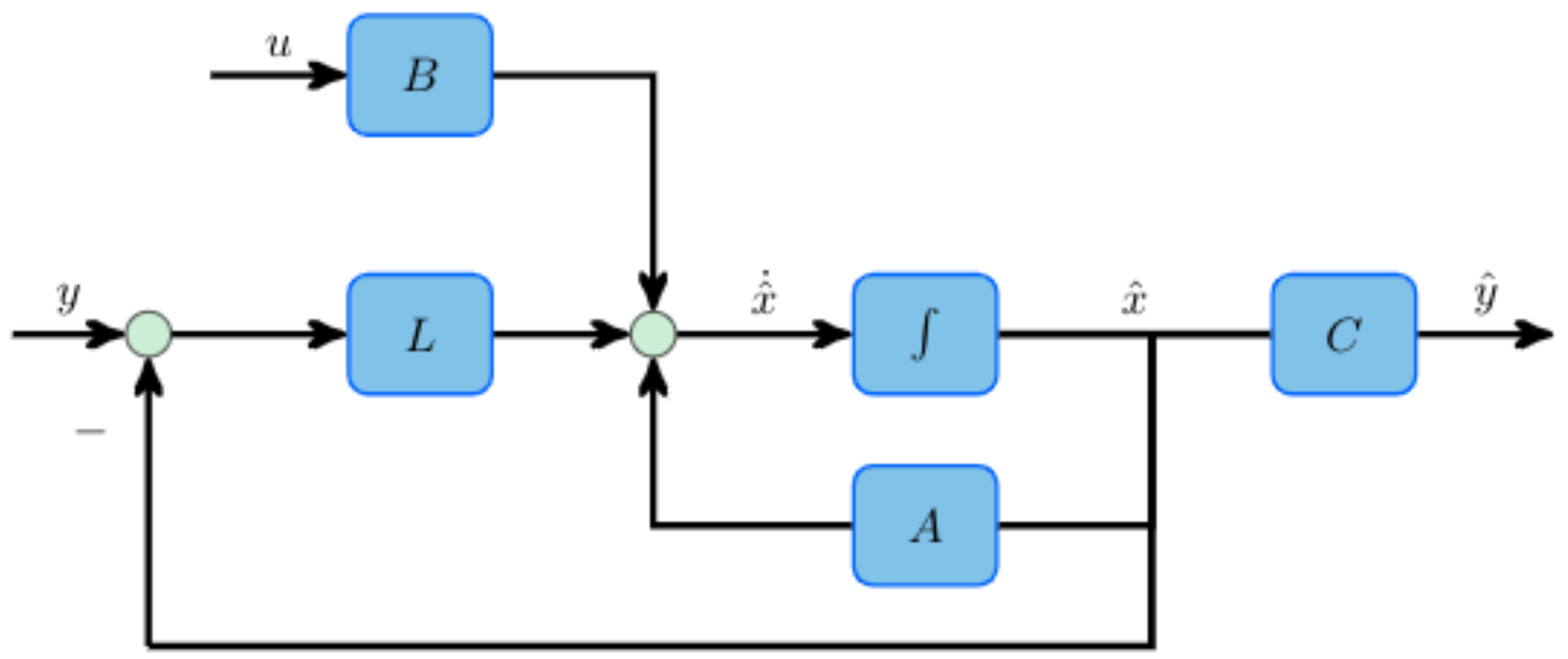
We can leverage similar synthesis tools as with stabilizability to synthesis what are known as observers. This amounts to designing an estimation scheme.

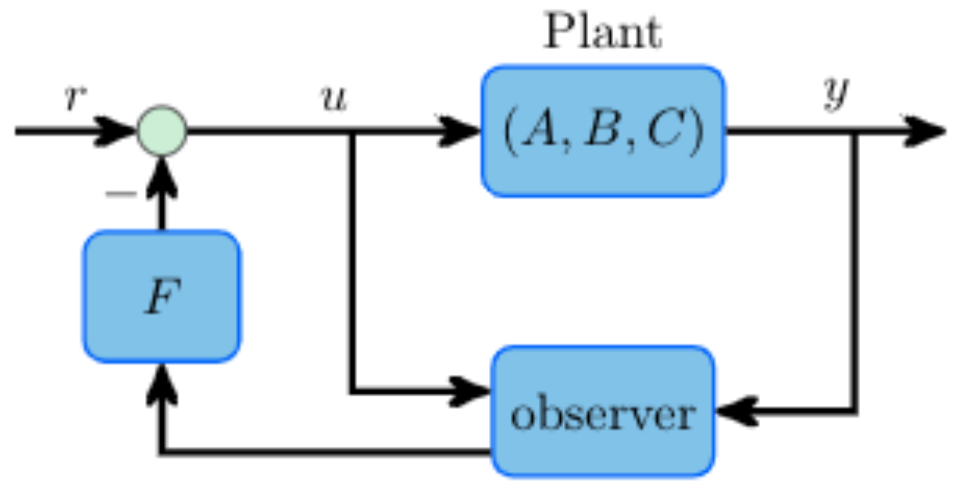
Observer Design

Consider the system $\dot{x} = Ax + Bu, \quad y = Cx$ and let $u = -Kx$ be a stabilizing feedback controller. When only the output y can be measured, the control law cannot be implemented, but if the pair (A, C) is detectable, it should be possible to estimate x from the system's output up to an error that vanishes as $t \to \infty$.

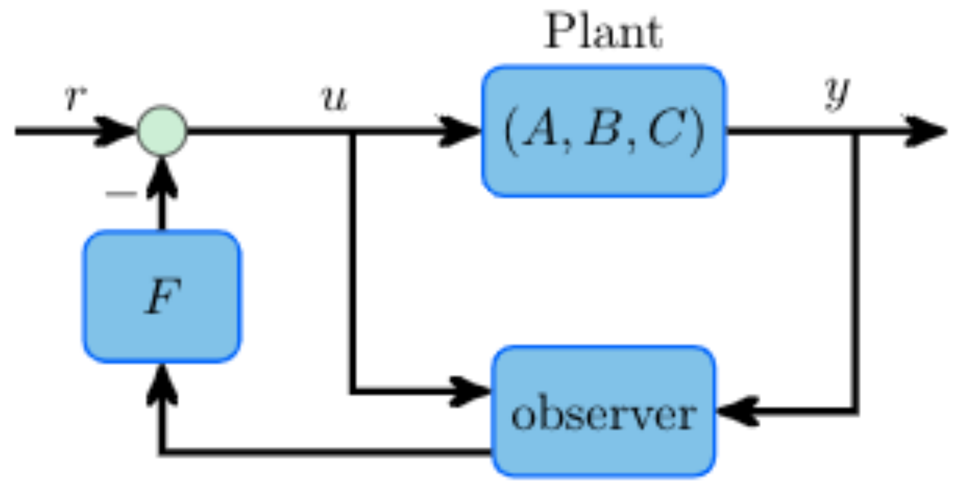


Observer Design





Observer Design: Example



Consider the system

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix} x(t)$$

Suppose we want to place the poles of the observer at $\{-4, -4\}$.