Mod3-RL4: Controllability & Observability of LTI Systems

References:

- Chapter 8 Callier & Desoer **[C&D]**
- Chapter 11 and 15 Hespanha **[JH]**
- **[510]** Lecture Notes (Finite Rank Operator Lemma, Hilbert Spaces, Adjoints, etc.)

LTI Systems

Consider the following LTI system:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Solution:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\ d\tau$$

Lecture Goals:

- Derive the reachability (controllability) and observability grammian for the LTI setting
- Develop various tests for controllability and observability

Controllability/Reachability Grammian

Reachability Grammian:

change of variables

$$W_r = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) B^*(\tau) \Phi^*(t_1, \tau)\ d\tau = \int_{t_0}^{t_1} e^{A(t_1-\tau)} B B^* e^{A^*(t_1-\tau)}\ d\tau = \int_0^{t_1-t_0} e^{At} B B^* e^{A^*t}\ dt$$

Analogously, Controllability Grammian:

$$W_c = \int_0^{t_1-t_0} e^{-At} B B^* e^{-A^*t}\ dt$$

Cayley Hamilton and the expression for the matrix exponential let us derive the so-called controllability matrix:

$$\mathscr{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{C}^{n\times nm}$$

Controllability matrix

$$\mathscr{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{C}^{n \times nm}$$

Proposition. The following equality holds: $\operatorname{Im}(W_r) = \operatorname{Im}(\mathscr{C})$

Observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{C}^{np \times n}$$

Proposition. The following equality holds: $\mathrm{Ker}(W_o) = \mathrm{Ker}(\mathcal{O})$

Controllability & Observability Tests

We can derive ``tests'' based on $\mathscr{C}$ and $\mathscr{O}$ in order to check observability and controllability properties of the LTI system (A, B, C). We will start with controllability test.

Theorem. Let $\Delta = t_1 - t_0$ for some $t_1 > t_0$. The following are equivalent:

$$\begin{aligned} (1)\quad (A,B) \text{ controllable on } [0,\Delta] &\iff \text{rank}([B\ AB\ \cdots\ A^{n-1}B]) = n \quad (2) \\ &\iff \text{rank}([sI - A\ \ B]) = n \quad \forall s \in \mathbb{C} \quad (3) \end{aligned}$$

(2): Controllability test

(3): PBH test for controllability (Popov-Belevitch-Hautus)

Proof Sketch

Proof Sketch

Controllability: Example

Observability Tests

Theorem. Let $\Delta = t_1 - t_0$ for some $t_1 > t_0$. The following are equivalent:

$$(1)\quad (A, C) \text{ observable on } [0,\Delta] \iff \text{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n \qquad (2)$$

$$\iff \text{rank}\left(\begin{bmatrix} sI - A \\ C \end{bmatrix}\right) = n, \quad \forall\, s \in \mathbb{C} \qquad (3)$$

Analogous to the proof by contradiction that the PBH test for controllability implies the rank test for controllability, we can prove (3) $\implies$ (2) by way of constructing the so called observable decomposition:

There exists a transformation of coordinates to the form

$$\dot{z} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} z + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} \tilde{C} & 0 \end{bmatrix} z$$

Lyapunov Tests for Controllability/Observability

We saw in the previous recorded lectures in this module that

$$t_1 \mapsto W_r \text{ is the solution to } \dot{X}(t) = A(t)X(t) + X(t)A^*(t) + B(t)B^*(t), \text{ with } X(t_0) = 0$$

and $$t_0 \mapsto W_o \text{ is the solution to } \dot{X}(t) = -A(t)X(t) - X(t)A^*(t) - C(t)^*C(t), \text{ with } X(t_1) = 0$$

For LTI this gives rise to the Lypunov tests for controllability and observability.

Proposition. Assume A is Hurwitz stable. The LTI system (A, B) is controllable if and only if there exists a unique positive definite solution W to the following Lyapunov equation

$$AW + WA^\top = -BB^\top$$

Moreover, the solution is

$$W = \int_0^\infty e^{A\tau}BB^\top e^{A^\top\tau}\, d\tau$$

There is an analogous result for observability: $AW + WA^\top = -C^\top C$

Why useful? Synthesis of feedback controllers that stabilize the system.

Feedback Stabilization

Proposition. Let $\chi_A(s) = \det(sI - A)$ denote the characteristic polynomial of A. Fro any monoic real polynomial π of degree n, there exists a matrix $F \in \mathbb{R}^{m \times n}$ such that $\chi_{A+BF} = \pi$ if and only if the pair (A, B) is controllable.

two methods:

- Lyapunov
- Controllable cannonical form

Feedback Stabilization based on Lyapunov

fact: (A, B) is controllable $\implies (-\mu I - A, B)$ is controllable for every $\mu \in \mathbb{R}$.

Goal: derive an expression for a feedback controller that stabilizes the system

Controllable Cannonical Form

Any completely controllable system can be transformed to the controllable cannonical form:

$$\left\{ \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \right.$$

Why is this useful? The benefit of having the system in this form is that its easy to assess stability and its easy to design a feedback controller to place the poles for example to stabilize the system.

For example, the characteristic polynomial is given by $\chi_A(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n$

and with feedback $u = -Kx$, $K = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \in \mathbb{R}^{1\times n}$, we have the closed loop system given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ -a_n - k_1 & -a_{n-1} - k_2 & -a_{n-2} - k_3 & -a_{n-3} - k_4 & \cdots & -a_1 - k_n \end{bmatrix} x$$

whose characteristic polynomial is

$$\chi_A(s) = s^n + (\alpha_1 + k_n)s^{n-1} + (\alpha_2 + k_{n-1})s^{n-1} + \cdots + (\alpha_{n-1} + k_2)s + (\alpha_n + k_1)$$

Example: Pole Placement