Mod3-RL2: Controllability of LTV Systems

References:

- Chapter 8.2-8.4 Callier & Desoer [C&D]
- Chapter 11 and 15 Hespanha [JH]
- [510] Lecture Notes (Finite Rank Operator Lemma, Hilbert Spaces, Adjoints, etc.)

LTV System and Solution Reminder

Consider the LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

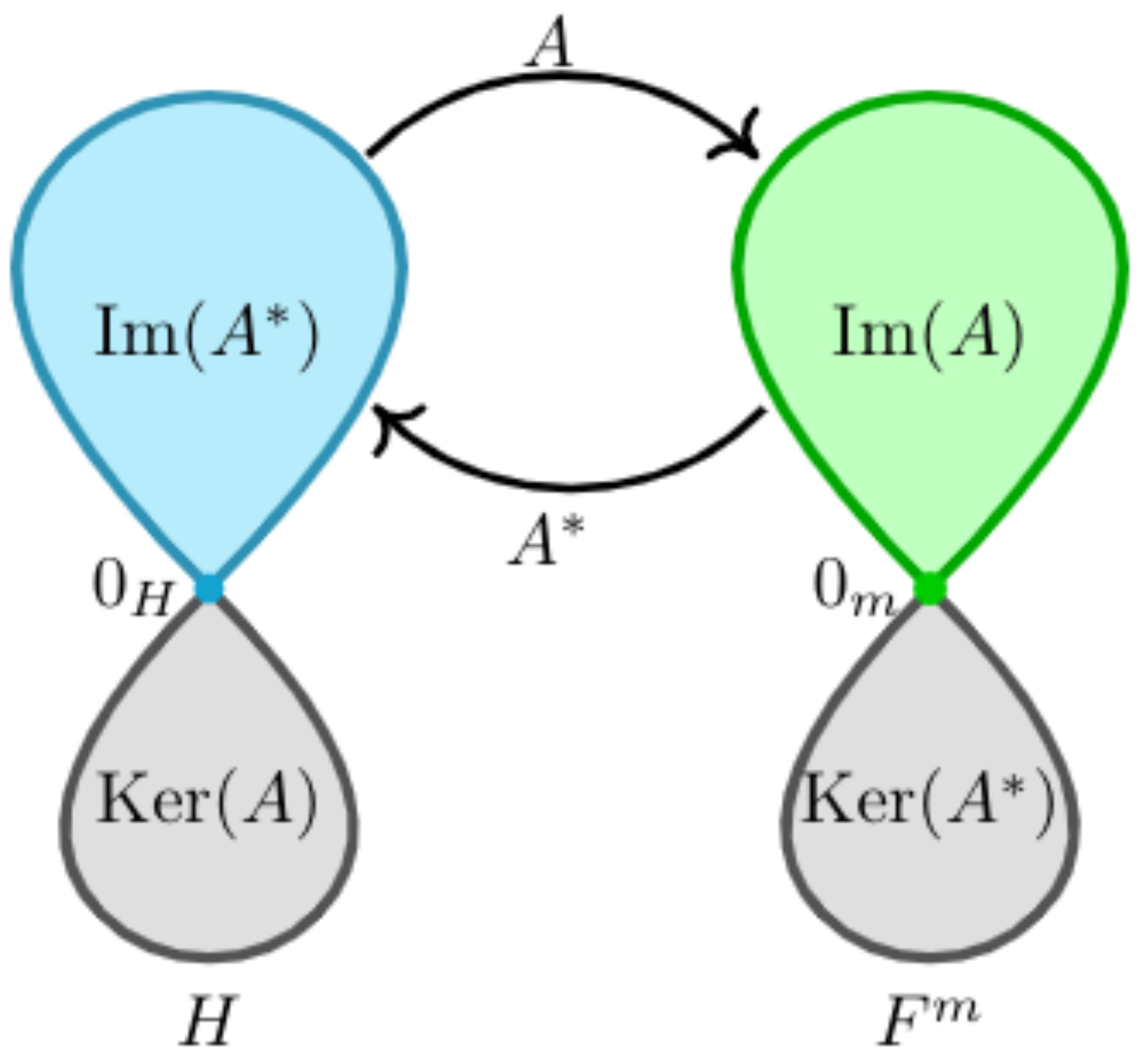
Solution is

$$x(t) = \phi(t, t_0, x_0, u_{[t_0,t_1]}) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)\ d\tau$$

Reminder [510]: Finite Rank Operator (FRO) Lemma

Lemma. Consider the linear map $A : H \to F^m$ from m-dimensional Hilbert space H to m-dimensional Hilbert space F^m (either $\mathbb{R}^m$ or $\mathbb{C}^m$). The following decompositions hold:

(a) $F^m = \mathrm{Im}(A) \overset{\perp}{\oplus} \mathrm{Ker}(A^*)$ (b) $H = \mathrm{Im}(A^*) \overset{\perp}{\oplus} \mathrm{Ker}(A)$



$$AA^* : F^m \to F^m \qquad A^*A : H \to H$$

Moreover, we have that

$$\mathrm{Ker}(AA^*) = \mathrm{Ker}(A^*), \qquad \mathrm{Im}(AA^*) = \mathrm{Im}(A)$$
$$\mathrm{Ker}(A^*A) = \mathrm{Ker}(A), \qquad \mathrm{Im}(A^*A) = \mathrm{Im}(A^*)$$

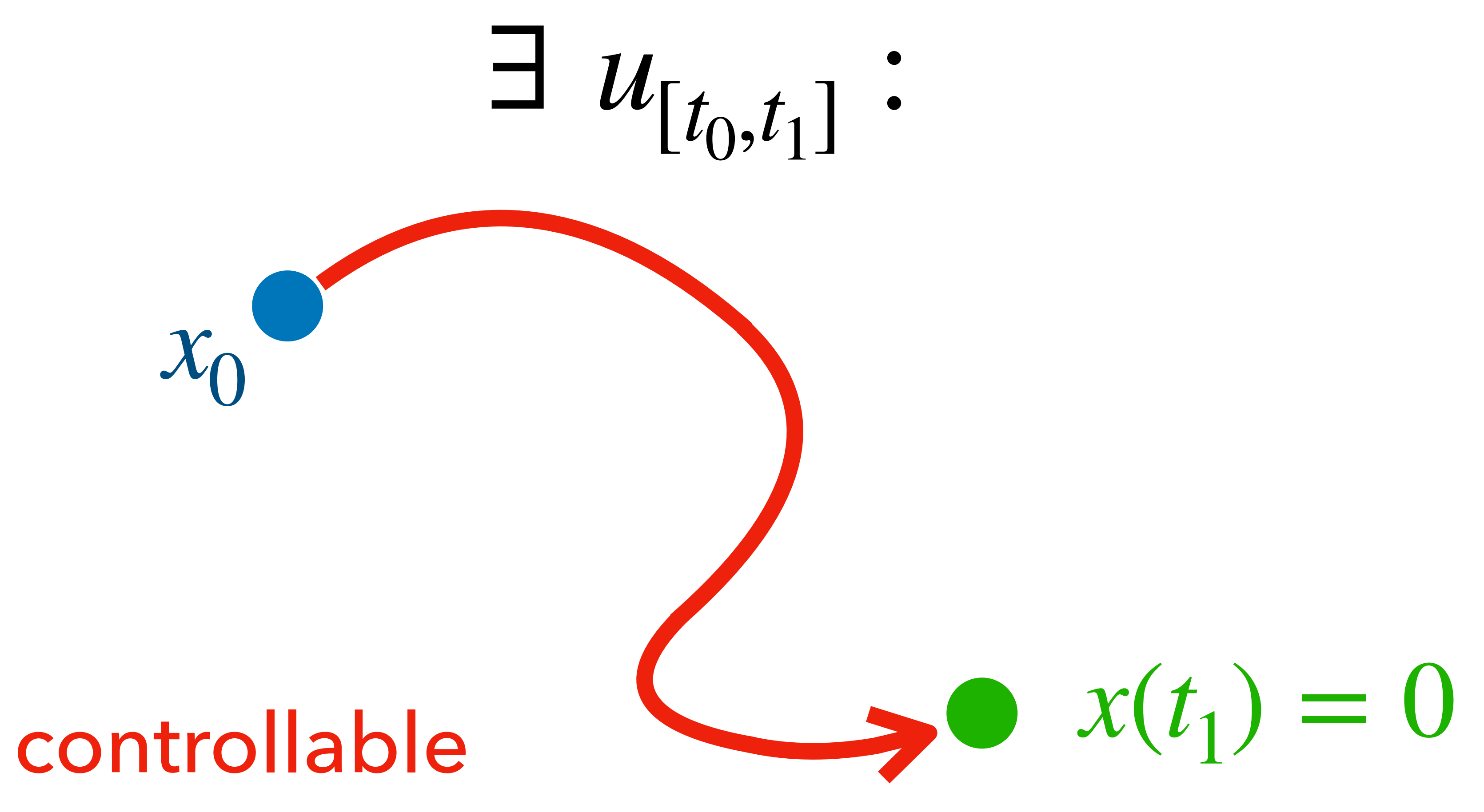
Controllability and Reachability of LTVs

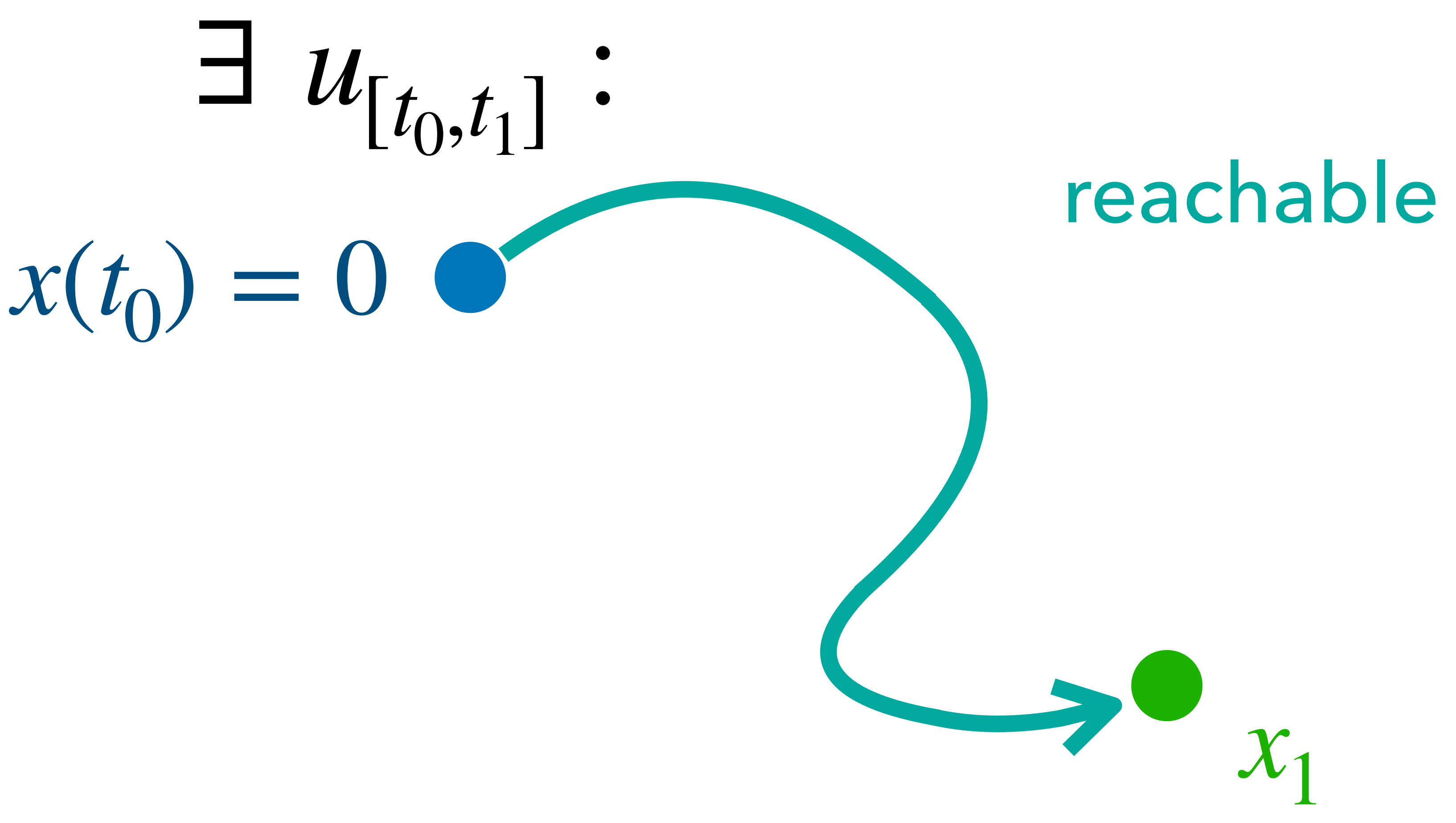
$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t)$$

short-hand: we will refer to controllability (reachability) of the pair $(A(\cdot), B(\cdot))$

Definition. Consider the pair $(A(\cdot), B(\cdot))$.

- The state x_0 is controllable to zero on $[t_0, t_1]$ if and only if there exists $u_{[t_0,t_1]}$ that steers (x_0, t_0) to $(0, t_1)$.
- The state x_1 is rechable (from the origin) on $[t_0, t_1]$ if and only if there exists $u_{[t_0,t_1]}$ that steers $(0, t_0)$ to (x_1, t_1).





Rechability Map

To define the reachability map, we consider the expression for $x(t_1)$:

Rechability Map

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$$
\begin{aligned}
x_1 := x(t_1) = \phi(t_1, t_0, x_0, u_{[t_0,t_1]}) &= \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau \\
&= \Phi(t_1, t_0)x_0 + \mathscr{L}_r u
\end{aligned}
$$

Hence, $\mathscr{L}_r : \mathrm{PC}([t_0, t_1]) \to \mathbb{C}^n$ is defined by

$$
\mathscr{L}_{r,[t_0,t_1]}(u(\cdot)) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau
$$

The expression for x_1 shows that there will be an input $u_{[t_0,t_1]}$ that transfers an arbitrary (x_0, t_0) to an arbitrary (x_1, t_1) if and only if the map $\mathscr{L}_{r,[t_0,t_1]} : \mathrm{PC}([t_0, t_1]) \to \mathbb{C}^n$ is surjective (onto).

Proposition. The following equivalence holds:

$$
\big(A(\cdot), B(\cdot)\big) \text{ is controllable on } [t_0, t_1] \iff \mathscr{L}_{r,[t_0,t_1]}(u(\cdot)) \text{ is surjective}
$$

Reachable subspace

The map $\mathscr{L}_r : \mathrm{PC}([t_0, t_1]) \to \mathbb{C}^n$ determines the set of states that can be reached from the origin on some time interval.

Adjoint of the Reachability Map

To understand controllability in terms of reachability we construct what is known as the reachability Grammian. To this end we need to first compute the adjoint of $\mathscr{L}_r : \mathscr{U}_{[t_0,t_1]} \to \mathbb{R}^n$.

$$\mathscr{L}_r(u(\,\cdot\,)) = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)\ d\tau$$

Claim. The adjoint is $\mathscr{L}_r^* x : B^*(\,\cdot\,)\Phi^*(t_1,\,\cdot\,)x$

proof.

Reachability Grammian

With the adjoint map in hand, we can define the reachability grammian $\mathscr{L}_r \mathscr{L}_r^* : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

Controllability in terms of reachability

With the adjoint map in hand, we can define the reachability grammian $\mathscr{L}_r\mathscr{L}_r^* : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\mathscr{L}_r\mathscr{L}_r^* = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B^*(\tau)\Phi^*(t_1, \tau)\ d\tau$$

Theorem. Let $(A(\cdot), B(\cdot))$ be piecewise continuous. Then, we have the following equivalences:

$$\begin{aligned} (A(\cdot), B(\cdot)) \text{ controllable on } [t_0, t_1] &\iff \mathrm{Im}(\mathscr{L}_r) = \mathbb{C}^n && (1) \\ &\iff \mathrm{Im}(\mathscr{L}_r\mathscr{L}_r^*) = \mathbb{C}^n && (2) \\ &\iff \det(W_{r,[t_0,t_1]}) \neq 0 && (3) \end{aligned}$$

Further, the set of reachable states on $[t_0, t_1]$ is the subspace $\mathrm{Im}(\mathscr{L}_r) = \mathrm{Im}(W_r)$, where we drop the interval subscript when clear from context.

Controllability in terms of reachability

Theorem. Let $(A(\cdot), B(\cdot))$ be piecewise continuous. Then, we have the following equivalences:

$$
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(A(\cdot), B(\cdot)) \text{ controllable on } [t_0, t_1] &\iff \mathrm{Im}(\mathscr{L}_r) = \mathbb{C}^n && (1)\\
&\iff \mathrm{Im}(\mathscr{L}_r\mathscr{L}_r^*) = \mathbb{C}^n && (2)\\
&\iff \det(W_{r,[t_0,t_1]}) \neq 0 && (3)
\end{aligned}
$$

proof sketch.

Controllability Map

The equivalence in the preceding theorem let's us derive the analog to the rechability map: controllability map

$$\mathscr{L}_c : u_{[t_0,t_1]} \mapsto \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\ d\tau$$

Controllability Map: Connection to Reachability Map

Proposition. The following equivalence holds:

$$\mathscr{L}_c \text{ is surjective} \iff \exists\ u_{[t_0,t_1]} \text{ that steers arbitrary } (x_0, t_0) \text{ to arbitrary } (x_1, t_1)\,.$$

Proposition. The reacbility map is related to the controllability map:

$$\mathrm{Im}(\mathscr{L}_r) = \Phi(t_1, t_0)\mathrm{Im}(\mathscr{L}_c)$$

Controllability Subspace and Grammian

With the adjoint map in hand, we can define the controllability grammian $\mathscr{L}_c\mathscr{L}_c^* : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\mathscr{L}_c\mathscr{L}_c^* = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B^*(\tau)\Phi^*(t_0, \tau)\ d\tau$$

Theorem. Let $(A(\cdot), B(\cdot))$ be piecewise continuous. Then, we have the following equivalences:

$$\begin{aligned} (A(\cdot), B(\cdot)) \text{ controllable on } [t_0, t_1] &\iff \mathrm{Im}(\mathscr{L}_c) = \mathbb{C}^n && (1) \\ &\iff \mathrm{Im}(\mathscr{L}_c\mathscr{L}_c^*) = \mathbb{C}^n && (2) \\ &\iff \det(W_{c,[t_0,t_1]}) \neq 0 && (3) \end{aligned}$$

Further, the controllable subspace is the subspace

$$\mathrm{Im}(\mathscr{L}_c) = \mathrm{Im}(W_c) = \left\{ x_0 \in \mathbb{C}^n :\ \exists u(\cdot),\ 0 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau \right\}$$

Mod3-RL2a: Application to Minimum Cost Control

References:

- Chapter 8.2-8.4 Callier & Desoer **[C&D]**
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Minimum Cost Control

One interesting application for the controllability/reachabiliy map and grammian is to the problem of finding the minimum cost control. Consider the cost of control to be givn by teh L_2-norm of $u(\cdot)$:

Minimum Cost Control

$$\tilde{u}(t) = B(t)^*\Phi(t_1, t)^* W_{r,[t_0,t_1]}^{-1}(x_1 - \Phi(t_1, t_0)x_0)$$

Minimum Cost Control: Effectiveness of the Actuators