Mod3-RL1: Intro to Controllability & Observability

References:

- Chapter 8/8d Callier & Desoer **[C&D]**
- Chapter 11 and 15 Hespanha **[JH]**
- **[510]** Lecture Notes (Finite Rank Operator Lemma, Hilbert Spaces, Adjoints, etc.)

Dynamical System Notation

The first component of this model seeks to introduce the basic fundamental concepts of controllability and observability of linear systems. The concepts are "dual" in some sense and hence we introduce them together.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

(LTV)

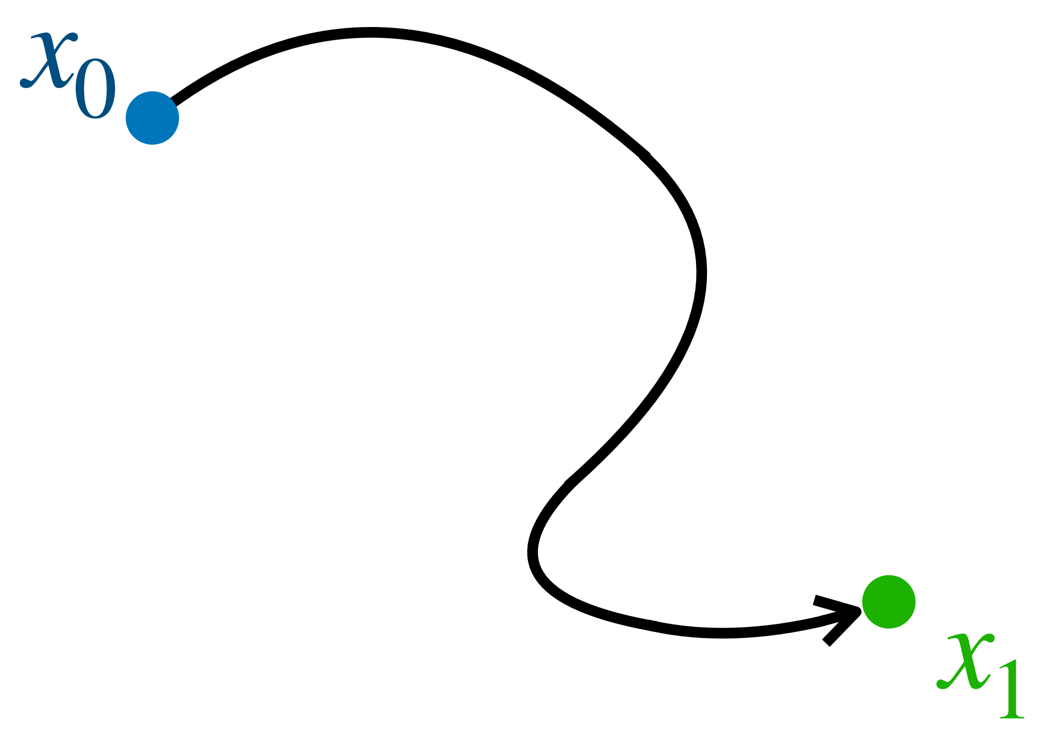
- state transition map (Flow): $\phi(t, t_0, x_0, u)$
- read out map (output): $\rho(t, t_0, x_0, u)$
- input space of piecewise continuous (PC) maps: $\mathscr{U}$
- state space of PC maps: $\mathscr{X}$
- output space of PC maps: $\mathscr{Y}$

$$\mathscr{D} = (\phi, \rho, \mathscr{U}, \mathscr{X}, \mathscr{Y})$$

Dynamical system

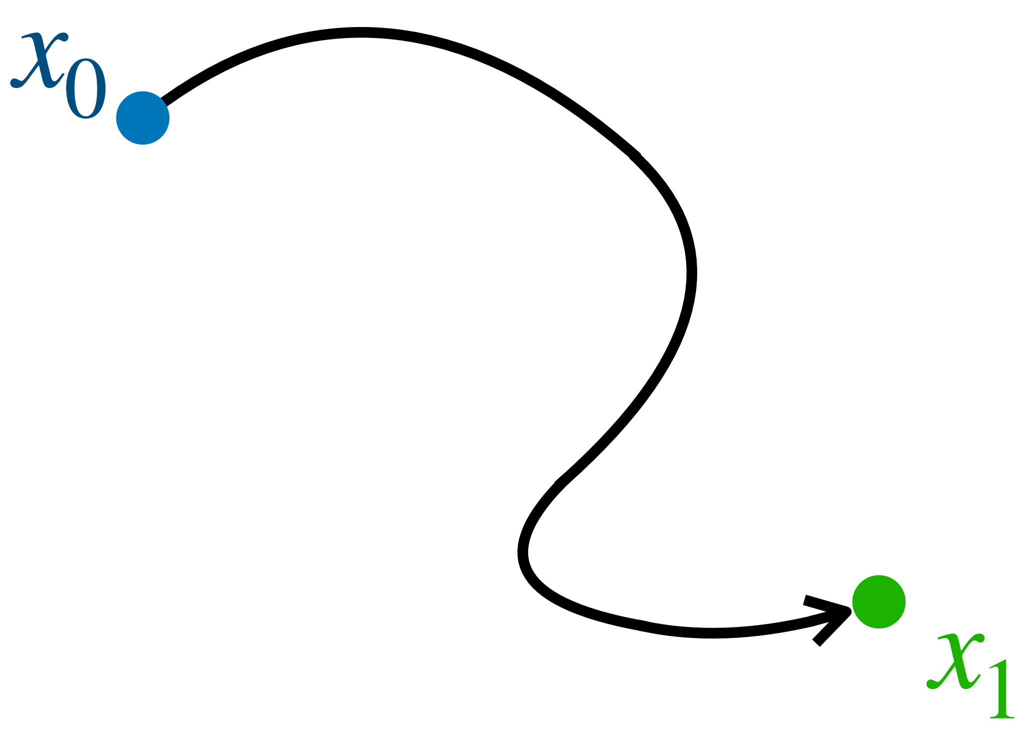
Notation: $x \in \mathscr{X}$ means $x : t \mapsto x(t) \in \mathbb{R}^n$

Steering



We say the input $u_{[t_0,t_1]}$ defined on $[t_0, t_1] \subset \mathbb{R}_+$ "steers" $x_0 := x(t_0)$ to $x_1 := x(t_1)$ if

Steering



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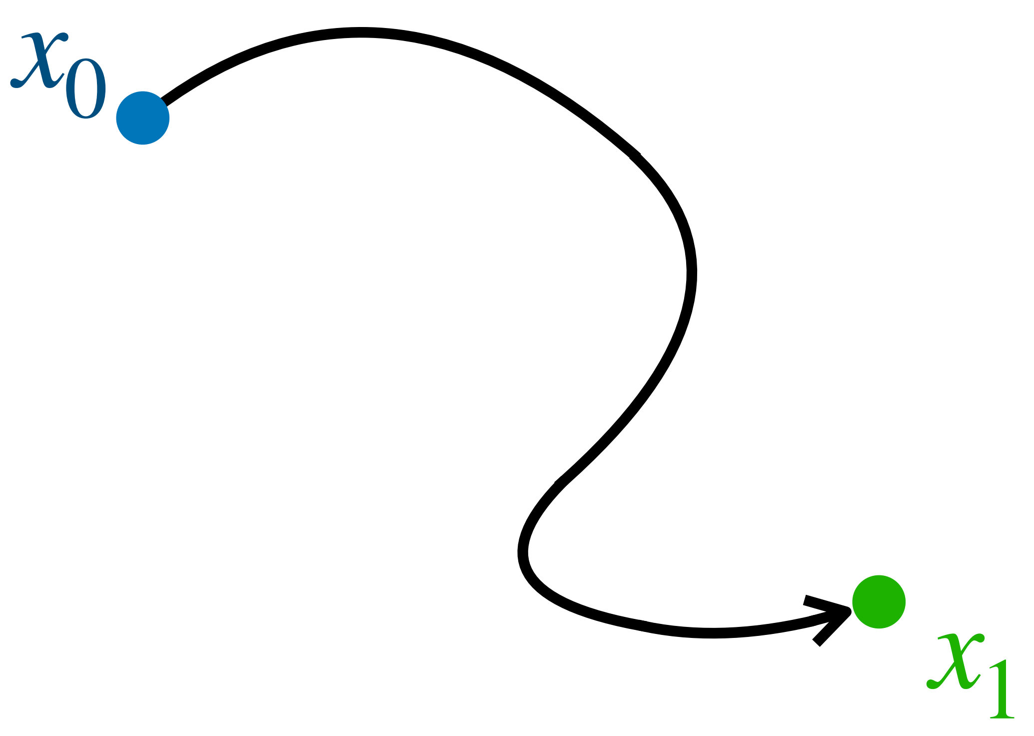
$$x_1 = \phi(t_1, t_0, x_0, u_{[t_0,t_1]}) = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau \in \mathbb{R}^n$$

Definition: (Controllable) The system representation $\mathscr{D}$ is **controllable** on $[t_0, t_1]$ if for all $(x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists $u_{[t_0,t_1]} \in \mathscr{U}$ which steers $x_0 = x(t_0)$ to $x_1 = x(t_1)$.

Steering



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We tend to break controllability into two different concepts

- controllability from the origin (reachability)
- controllability to the origin (often simply referred to as controllability)

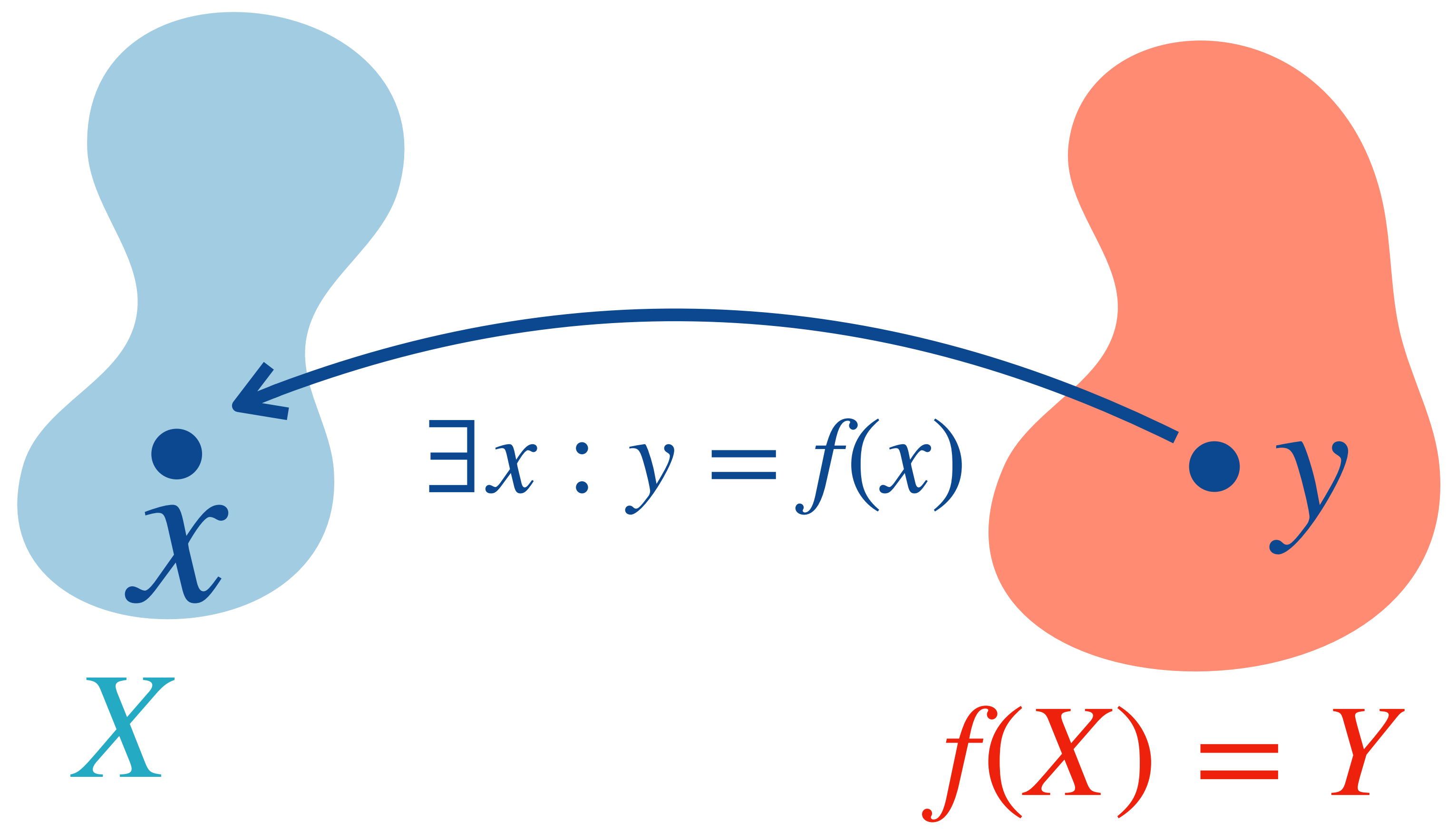
} if a system is both of these, it is completely controllable

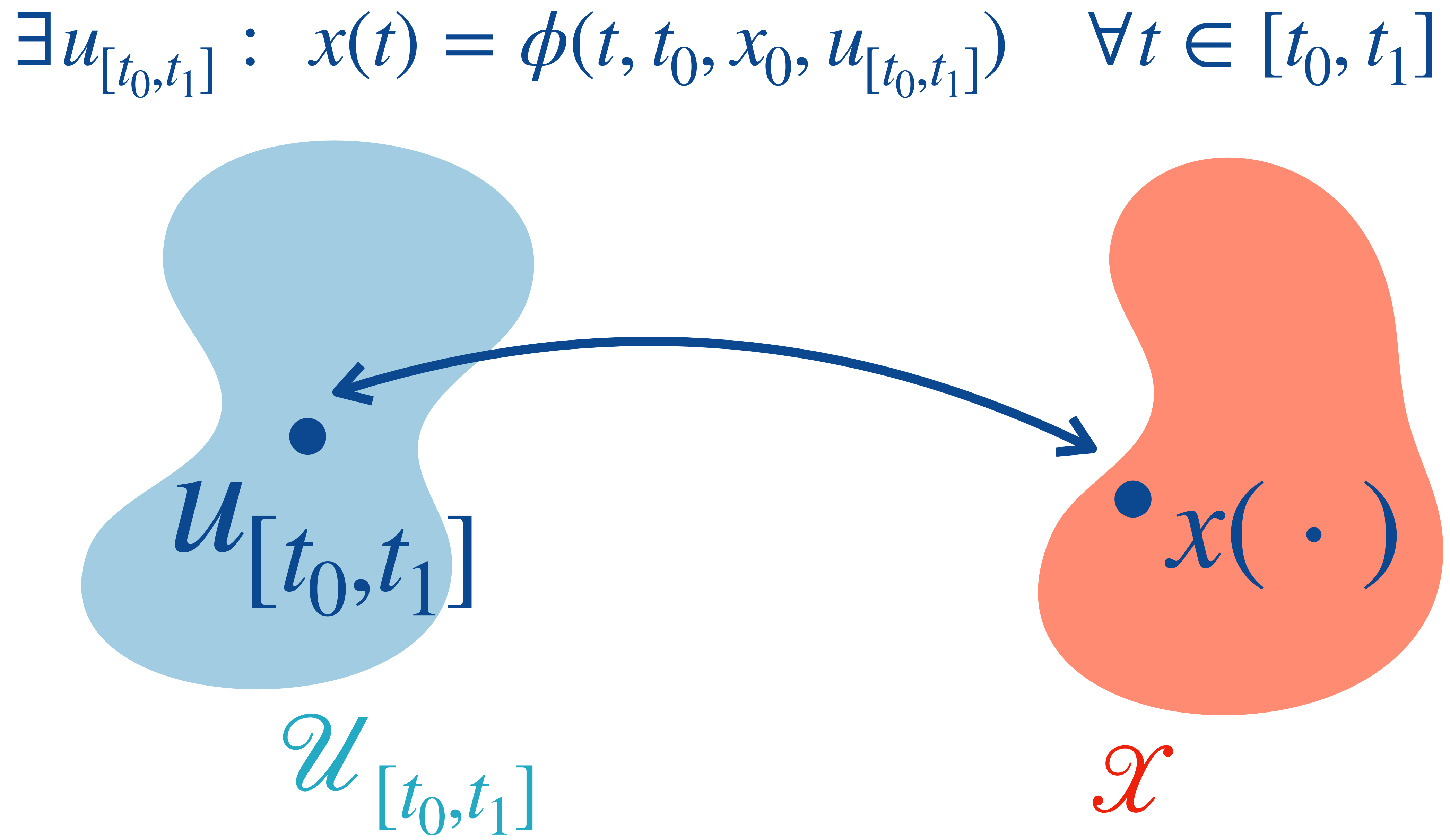
Controllability and Surjectivity

Reminder from [510]

Definition: a map $f : X \to Y$ is surjective (onto) if and only if

$$\forall y \in Y, \exists x \in X, \text{ such that } y = f(x)\,.$$





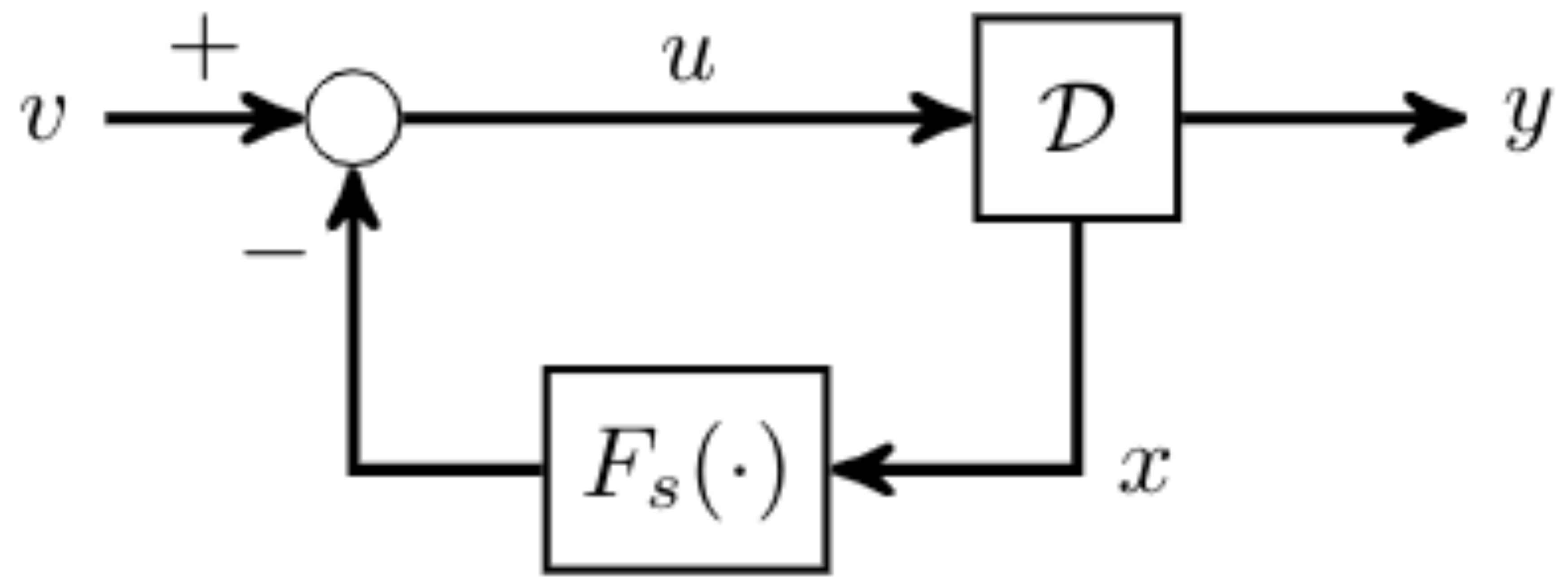
Proposition: The dynamical system $\mathscr{D}$ is controllable on $[t_0, t_1] \iff \forall\, x_0 \in \mathbb{R}^n$ the map

$$\phi(t_1, t_0, x_0, \cdot\,) : u_{[t_0,t_1]} \mapsto x(t)$$

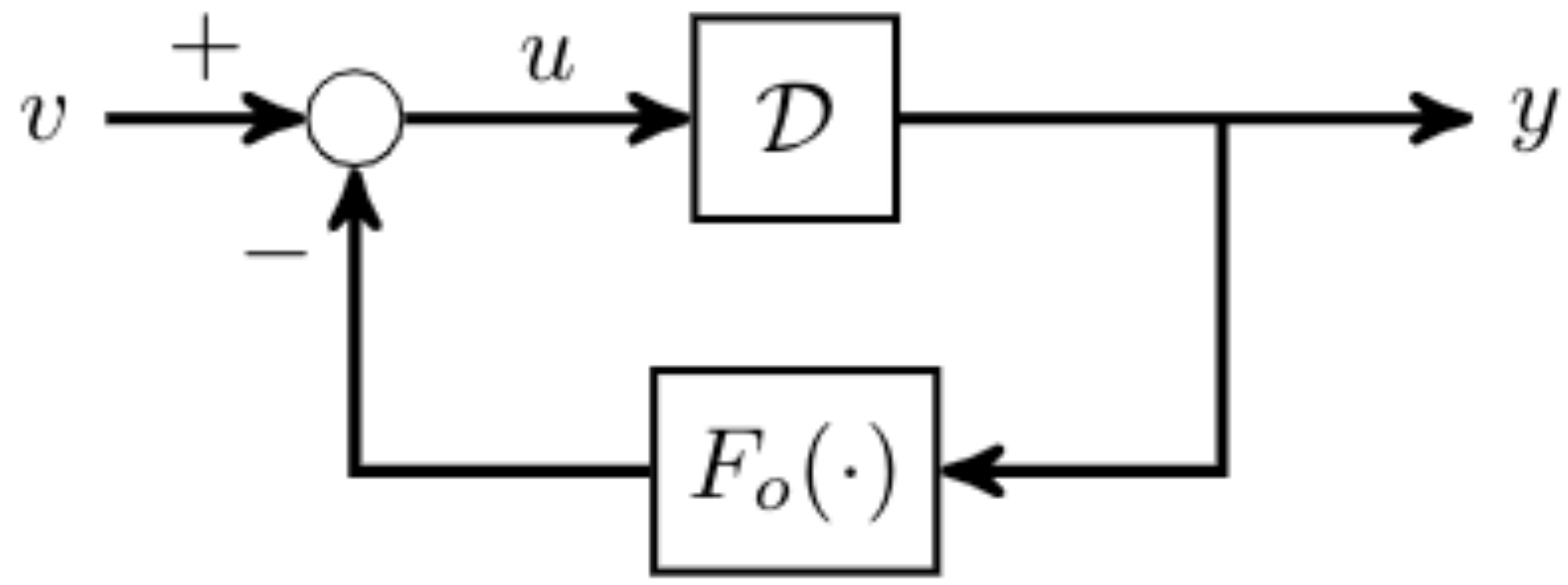
is surjective. That is, it maps $\mathscr{U}_{[t_0,t_1]}$ onto $\mathscr{X}$.

Memoryless Feedback and Controllability

memoryless state feedback



memoryless output feedback



Objective: understand how feedback control impacts controllability of the closed loop system

Controllability if Preserved Under Memoryless Feedback

Theorem. Let $\mathscr{D}_s$, $\mathscr{D}_o$ be well-posed. Then the following equivalences hold:

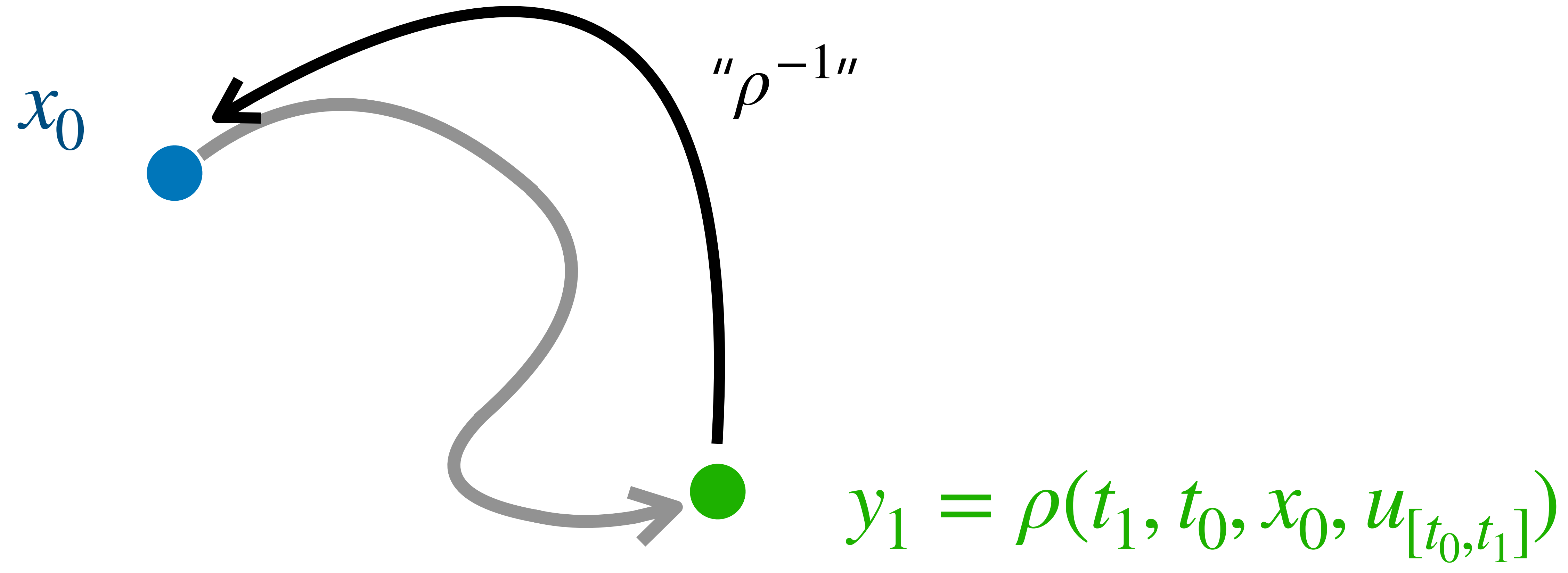
$$\begin{aligned} &\mathscr{D} \text{ is controllable on } [t_0, t_1] && \text{(a)} \\ \iff\ &\mathscr{D}_s \text{ is controllable on } [t_0, t_1] && \text{(b)} \\ \iff\ &\mathscr{D}_o \text{ is controllable on } [t_0, t_1] \end{aligned}$$

proof (a) $\iff$ (b)

Observability

Observability: characterizes under what conditions we can "observe" the state of the dynamical system given the output map

Definition. The dynamical system $\mathscr{D}$ is called observable on $[t_0, t_1]$ if and only if given, $\mathscr{D}$, for all inputs $u_{[t_0,t_1]}$ and for all corresponding outputs $y_{[t_0,t_1]} \in \mathscr{Y}$ the state x_0 at time t_0 is uniquely determined.

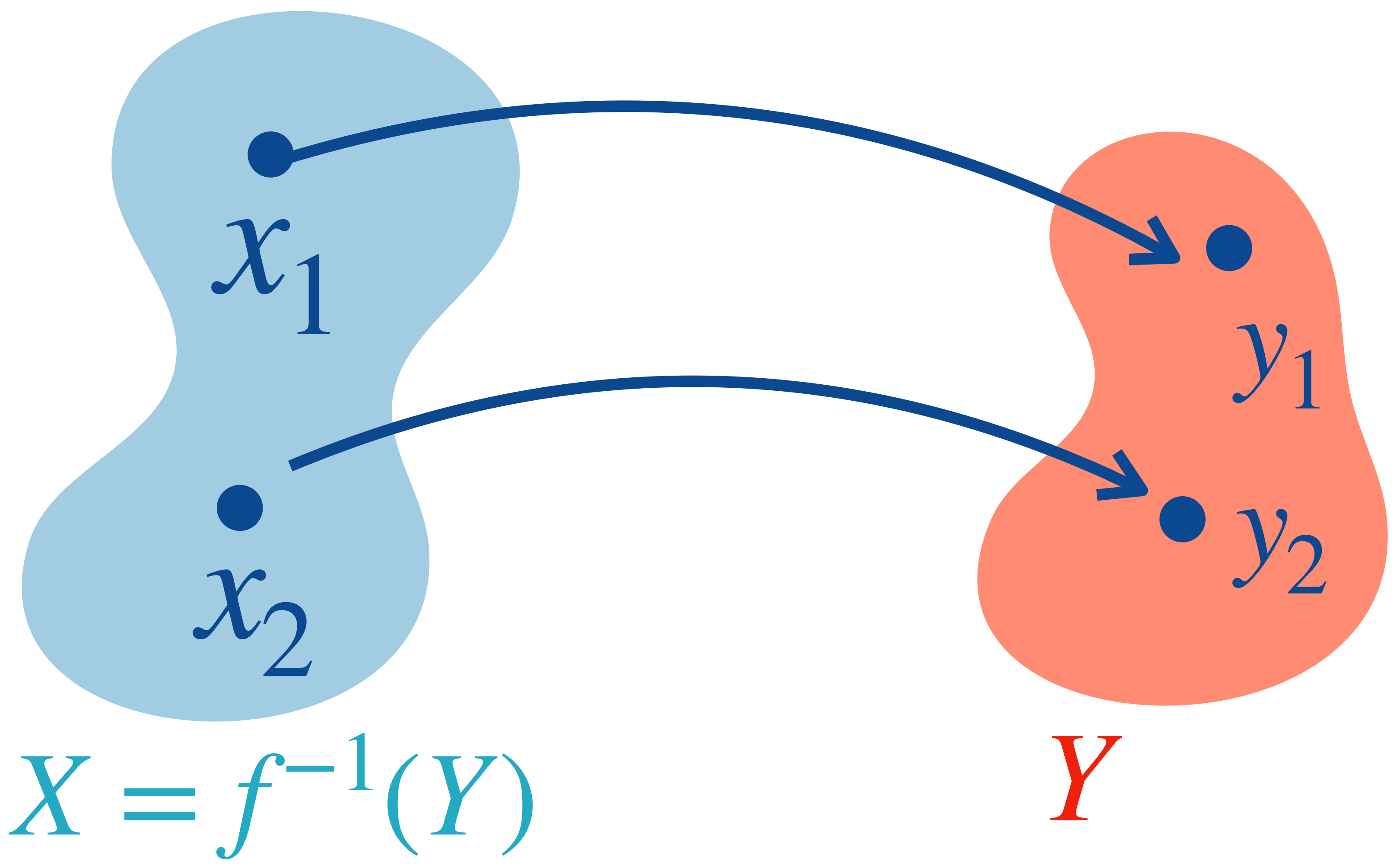


Observability and Injectivity

Reminder from [510]

Definition: a map $f : X \to Y$ is injective (one-to-one) if and only if

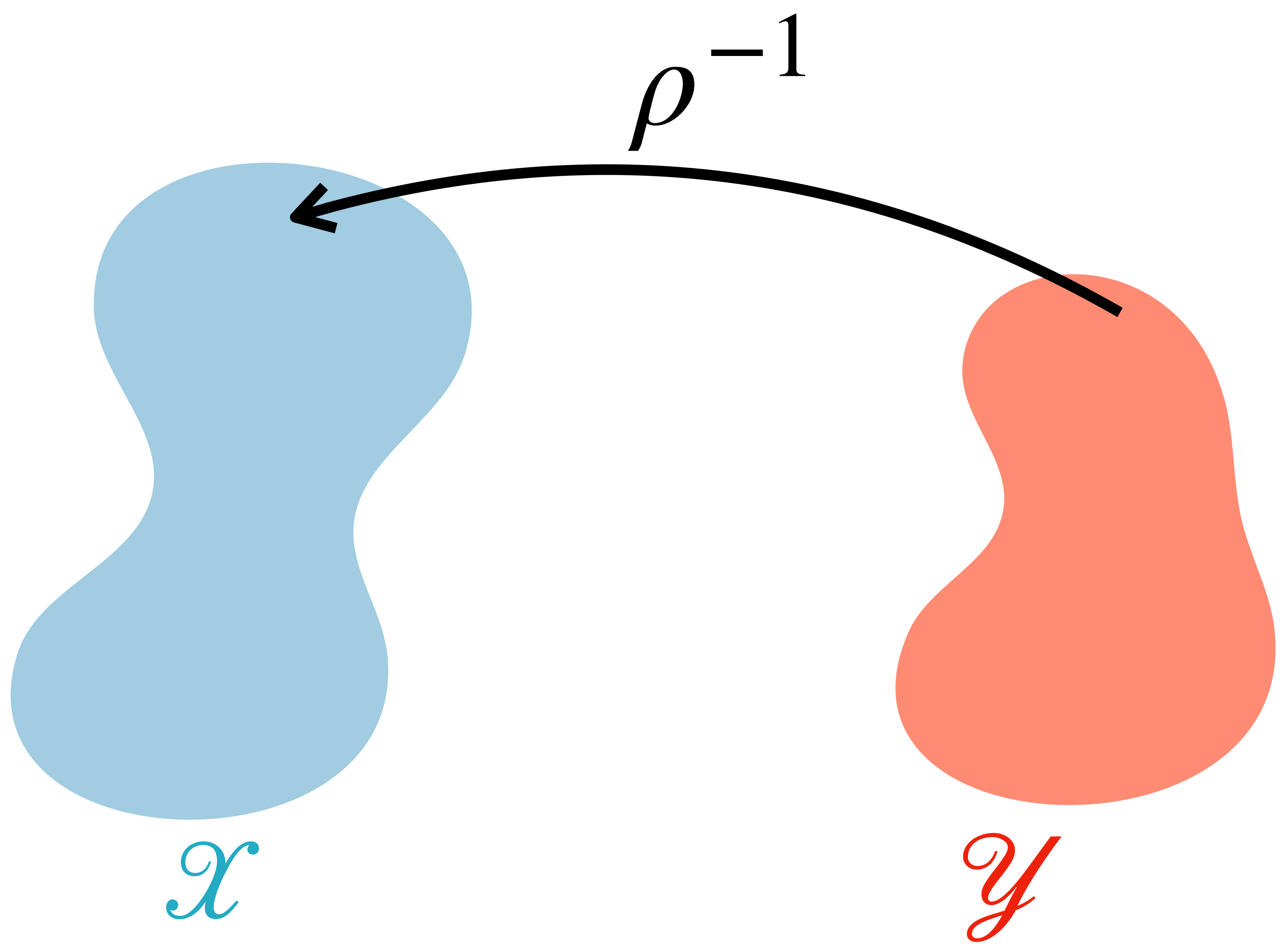
$$[f(x_1) = f(x_2) \implies x_1 = x_2] \iff [x_1 \neq x_2 \implies f(x_1) \neq f(x_2)].$$



Proposition. The dynamical system $\mathscr{D}$ is called observable on $[t_0, t_1]$ if and only if for each fixed input $u_{[t_0,t_1]}$, the partial response map

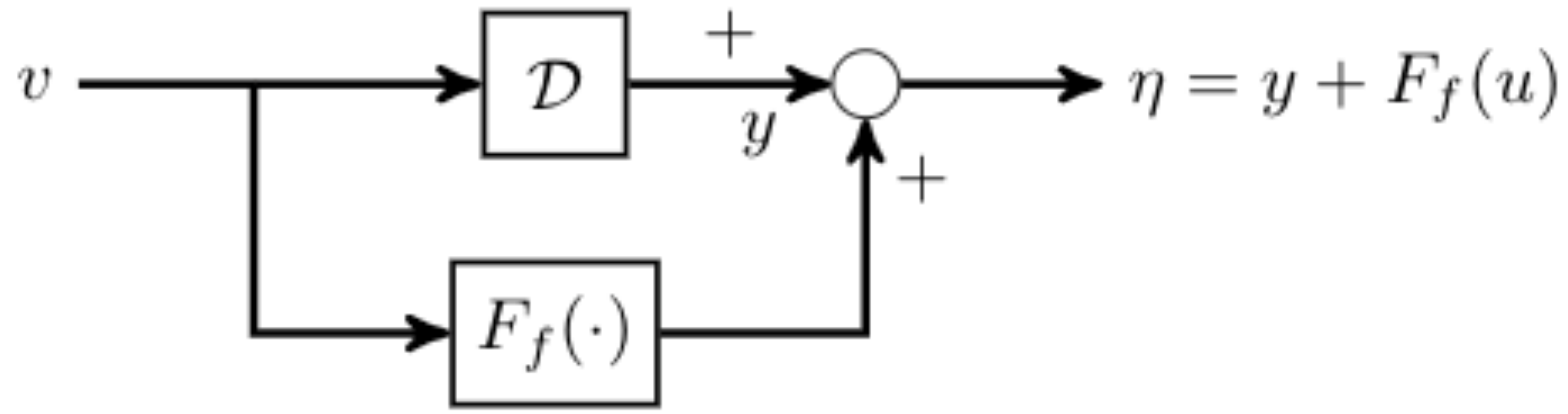
$$x_0 \mapsto y_{[t_0,t_1]} = \rho(\,\cdot\,, t_0, x_0, u_{[t_0,t_1]})$$

is injective; that is, the partial response map is a one-to-one map from $\mathscr{X}$ to $\mathscr{Y}$.

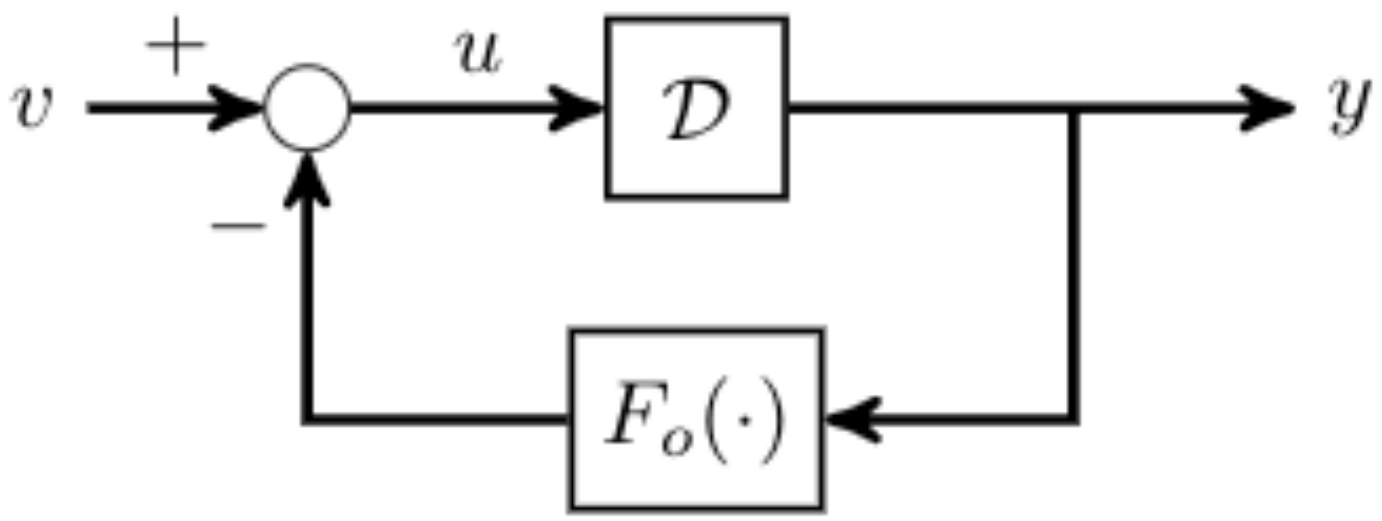


Memoryless Output Feedback & Feedforward Control

memoryless feedforward control



memoryless output feedback



Objective: understand how feedback and feedforward control impacts observability of the closed loop system

Observability is Preserved

Theorem. Let $\mathscr{D}_s$, $\mathscr{D}_o$ be well-posed. Then the following equivalences hold:

$$\begin{aligned} &\mathscr{D} \text{ is observable on } [t_0, t_1] \\ &\iff \mathscr{D}_o \text{ is observable on } [t_0, t_1] \\ &\iff \mathscr{D}_f \text{ is observable on } [t_0, t_1] \end{aligned}$$

Remark. **Memoryless state feedback may affect observability.** For example, for a linear time-invariant system representation (A, B, C, D), there may exist a linear state feedback F_s such that for some states x_0 and for some inputs $u(\cdot)$, the state trajectory remains in the nullspace of C for all t.