Mod2-RL3: Stability via Lyapunov's Equation

References:

- Chapter 4 and 7 Callier & Desoer **[C&D]**
- Chapter 8 and 9 Hespanha **[JH]**
- **[510]** Lecture Notes

Lyapunov Functions

The existance of a Lyapunov function is one way to prove stability. It essentially provides us with a certificate.

Definition: A function $V(x)$ is said to be positive definite if $V(z) \geq 0$, all sublevel sets are bounded, and

$$V(x) = 0 \iff x = 0$$

An α-sublevel set is defined by $\{x \in \mathbb{R}^n |\ V(x) \leq \alpha\}$. Sublevel sets benig bounded means that $V(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Definition: For a dynamical system $\dot{x} = f(x)$ with $x^* = 0$ as an equilibrium point, a scalar function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a Lyapunov function if it is C^1, locally positive definite, and $\dot{V} \leq 0$ where

$$\dot{V}(x) = \frac{d}{dt}V(x(t)) = \frac{\partial V}{\partial x}\frac{d}{dt}x = D_xV(x)f(x)$$

The existance of a Lyapunov function enables us to certify that all trajectories of the system reamin bounded, and hence the system is stable.

Lyapunov Functions: Asymptotic Stability

Consider the dynamical system $\dot{x} = f(x)$ with $x^* = 0$ as an equilibrium point. Suppose there is a scalar function $V : \mathbb{R}^n \to \mathbb{R}$ that is C^1, locally positive definite, and $\dot{V} < 0$ for all $x \neq 0$ and $\dot{V}(0) = 0$.

Then every trajectory of $\dot{x} = f(x)$ converges to zero as $t \to \infty$ meaning that the system is globally asymptotically stable.

Interpretation:

- the function V is a positive definite generalized energy function
- and energy is always dissipated, except at zero

Lyapunov Functions: Example

Consider the dynamical system
$$\dot{x}_1 = -x_1 + g(x_2)$$
$$\dot{x}_2 = -x_2 + h(x_1)$$
where $|g(z)| \leq |z|/2$ and $|h(z)| \leq |z|/2$.

Show that $V(z) = \frac{1}{2}\|x\|_2^2$ is a Lyapunov function.

Proof:

- Clearly V is positive definite
- Next we bound $\dot{V}$

Lyapunov Functions for LTI Systems

For linear systems, the case is much easier since it turns out that $V(x) = x^\top P x$ is always a Lyapunov function if one exists. That is the form of the Lyapunov function for stable linear system is this quadratic form.

For the linear system $\dot{x} = Ax$, if it is stable then we will be able to construct a $P = P^\top \succeq 0$ such that $V(x) = x^\top P x$ satisfies the requirements for a Lyapunov function. That is, it is positive definite and decreasing along trajectories.

Lyapunov Functions for LTI Systems

Lyapunov Theorem for CT LTI Systems

Theorem: The following conditions are equivalent.

a. The system $\dot{x} = Ax$ is asymptotically (equiv. exponentially) stable.
b. All eigenvalues of A have strictly negative real parts.
c. For every $Q = Q^\top \succ 0$, there exists a unique $P = P^\top \succ 0$ to the Lyapunov equation $A^\top P + PA = -Q$. Moreover, P is given by

$$P = \int_0^\infty e^{A^\top t} Q e^{At} \, dt$$

d. There exists a $P = P^\top \succ 0$ for which the following Lyapunov matrix inequality holds:

$$A^\top P + PA < 0$$

Lyapunov Theorem for LTI Systems

Theorem: The following conditions are equivalent.

a. The system $\dot{x} = Ax$ is asymptotically (equiv. exponentially) stable.
b. All eigenvalues of A have strictly negative real parts.
c. For every $Q = Q^\top \succ 0$, there exists a unique $P = P^\top \succ 0$ to the Lyapunov equation $A^\top P + PA = -Q$. Moreover, P is given by

$$P = \int_0^\infty e^{A^\top t} Q e^{At} \, dt$$

d. There exists a $P = P^\top \succ 0$ for which the following Lyapunov matrix inequality holds:

$$A^\top P + PA < 0$$

Proof Sketch for b $\implies$ c

Proof Sketch for d $\Longrightarrow$ b

Let $P = P^\top \succ 0$ such that $A^\top P + PA < 0$ and define $Q = -(A^\top P + PA)$

For an arbitrary solution $x(t)$ of the LTI system define the scalar time dependent map $v(t) = x^\top(t)Px(t) \geq 0$

Proof Sketch for d $\implies$ b: Comparison Lemma

Lemma: (Comparison Lemma) Let $v(t)$ be a differentiable scalar function. For some $\mu \in \mathbb{R}$, we have the following implication:

$$\dot{v} \leq \mu v(t), \;\; \forall \; t \geq t_0 \quad \implies \quad v(t) \leq e^{\mu(t-t_0)} v(t_0), \;\; \forall \; t \geq t_0$$

Applying the Lemma we have

Lyapunov Theorem for DT LTI Systems

Theorem: The following conditions are equivalent.

a. The system $x^+ = Ax$ is asymptotically (equiv. exponentially) stable.
b. All eigenvalues of A have modulus less than one.
c. For every $Q = Q^\top \succ 0$, there exists a unique $P = P^\top \succ 0$ to the Stein equation $A^\top PA - P = -Q$.
d. There exists a $P = P^\top \succ 0$ for which the following Lyapunov matrix inequality holds:

$$A^\top PA - P \prec 0$$

Using the Lyapunov Equation to Assess Performance

We can use the Lyapunov equation to assess **state feedback.**

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Suppose that the **closed loop dynamics** are stable. Then to evaluate natural quadratic integral performance measures we can use the Lyapunov equation.

$$\underbrace{J_u = \int_0^\infty u(t)^\top u(t)\ dt,}_{\text{energy of input}} \qquad \underbrace{J_y = \int_0^\infty y(t)^\top y(t)\ dt}_{\text{energy of output}}$$

Using the Lyapunov Equation to Assess Performance

Using the Lyapunov Equation to Assess Performance