Mod2-RL2: Spectral Conditions for Stabiliy

References:

- Chapter 4 & 7Callier & Desoer **[C&D]**
- Chapter 8 & Chapter 9 Hespanha **[JH]**
- Review [510] Lectures Notes on norms

LTI Systems

$$\dot{x} = Ax$$

- Stability of LTI systems (both CT and DT) reduces to checking the spectral properties of the system matrix A
- Intuition: From [510], we know how to compute functions of matrices. In particular, we know that

$$\exp(tA) = \sum_{k=1}^{p} \sum_{\ell=0}^{m_k-1} t^{\ell} \exp(\lambda_k t) p_{k\ell}(A)$$

where $p_{k,\ell}$ is a polynomial derived from the minimal polynomial, and m_k is the ascent of $A - \lambda_k I$.

Spectral Conditions for Stability

Proposition CT: $\dot{x} = Ax$ is exponentially stable $\iff$ $\sigma(A) \subset \mathbb{C}_-^\circ$

$$\exp(tA) = \sum_{k=1}^{p} \sum_{\ell=0}^{m_k-1} t^\ell \exp(\lambda_k t) p_{k\ell}(A)$$

Spectral Conditions for Stability

Proposition DT: $x^+ = Ax$ is exponentially stable $\iff$ $\sigma(A) \subset \mathbb{D}_1$

$$\forall\ \nu \in \mathbb{N},\ A^\nu = \sum_{k=1}^{p} \sum_{\ell=1}^{m_k-1} \nu(\nu-1)\cdots(\nu-\ell+1)\lambda_k^{\nu-\ell} p_{k\ell}(A$$

Applications

Two applications to better understand spectral stability properties:

1. M2-RL2a: Numerical Integration: Choosing stepsize to ensure convergence/stability
2. M2-RL2b: Nonlinear system stability through linearization and Hartman Grobman

M2-RL2a: Numerical Integration

Consider $\dot{x} = As,\ x(0) = x_0,\ A \in \mathbb{C}^{n\times n},\ \ x \in \mathbb{C}^n$

Call $t \mapsto x(t) = \exp(At)x_0$ the exact solution such that $t \mapsto x(t)$ is analytic in t.
Let $\xi_0, \xi_1, \ldots$ be the computed values at different times.

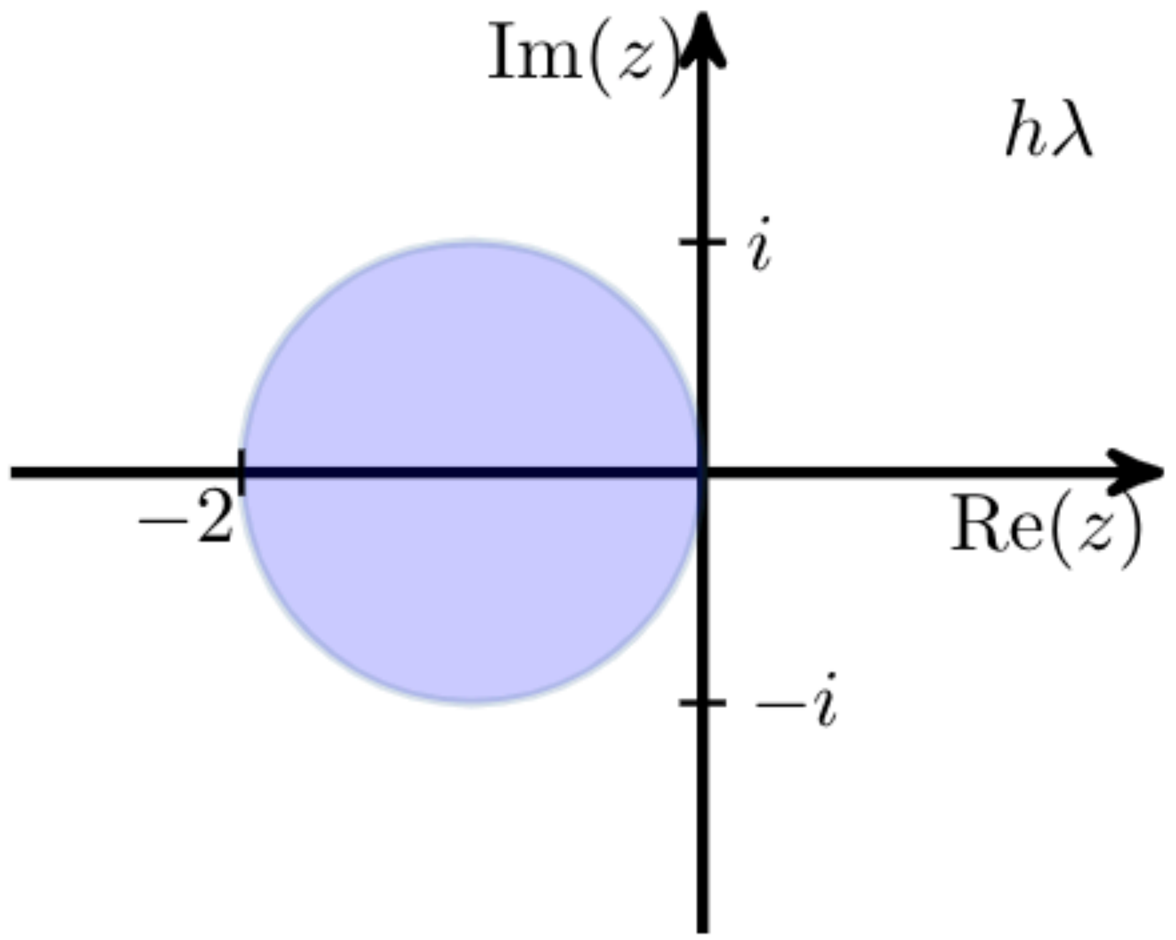
First Order Integration Schemes:

- Forward Euler
- Backward Euler

Forward Euler

Forward Euler: Example

Forward Euler: Example

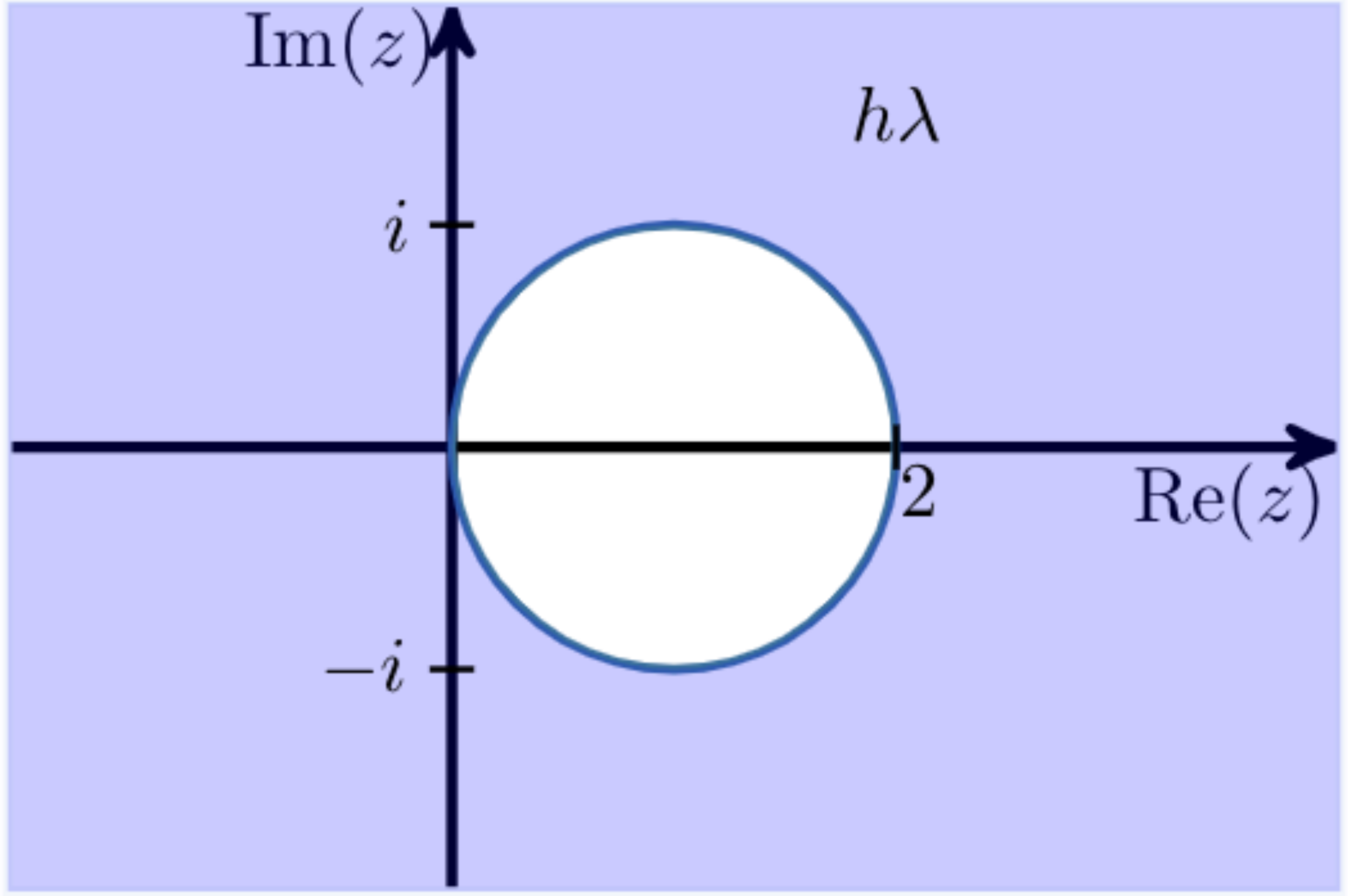


Forward Euler

Forward Euler: Convergence Criteria

Backward Euler

Backward Euler: Example



Backward Euler

Backward Euler: Convergence Criteria

Summary

- Convergence of numerical schemes for discretization go hand in hand with stability of LTI systems
- Picking a stepsize such that the numerical scheme converges to the exact solution requires assessing the spectrum of the dynamics

M2-RL2b: Stability of Nonlinear Systems via Linearization

- We saw in M2-RL1 that we can linearize a nonlinear system around an equilibrium point and assess stability of that nonlinear system in a neighborhood of the equilibrium via the linearized system.
- Take-Away: Linearization is effective in predicting qualitative patterns of behavior.
- The alternative is to construct a Lyapunov function, but this can be difficult for nonlinear systems (the notes discuss this and we will come back to this concept for LTI systems in the next lecture)

Theorem: [Hartman Grobman] Consider a nonlinear dynamical system $\dot{x} = f(x)$ with an equilibrium point x^* (i.e. $f(x^*) = 0$). If the linearization of the system $A := D_x f(x)|_{x=x^*}$ has no zero or purely imaginary eigenvalues then there exissts a homeomorphism (i.e., a conitnuous map with a continuous inverse) from a neighborhood U of x^* into $\mathbb{R}^n$,

$$h : U \to \mathbb{R}^n$$

taking trajectories of the nonlinear system $\dot{x} = f(x)$ and mapping them onto those of $\dot{\tilde{x}} = A\tilde{x}$. In particular, $h(x^*) = 0$.

Easier way to understand HG

linearization

$$\dot{x} = f(x) \quad \longrightarrow \quad \dot{\tilde{x}} = A\tilde{x},\ A := D_x f(x^*)$$

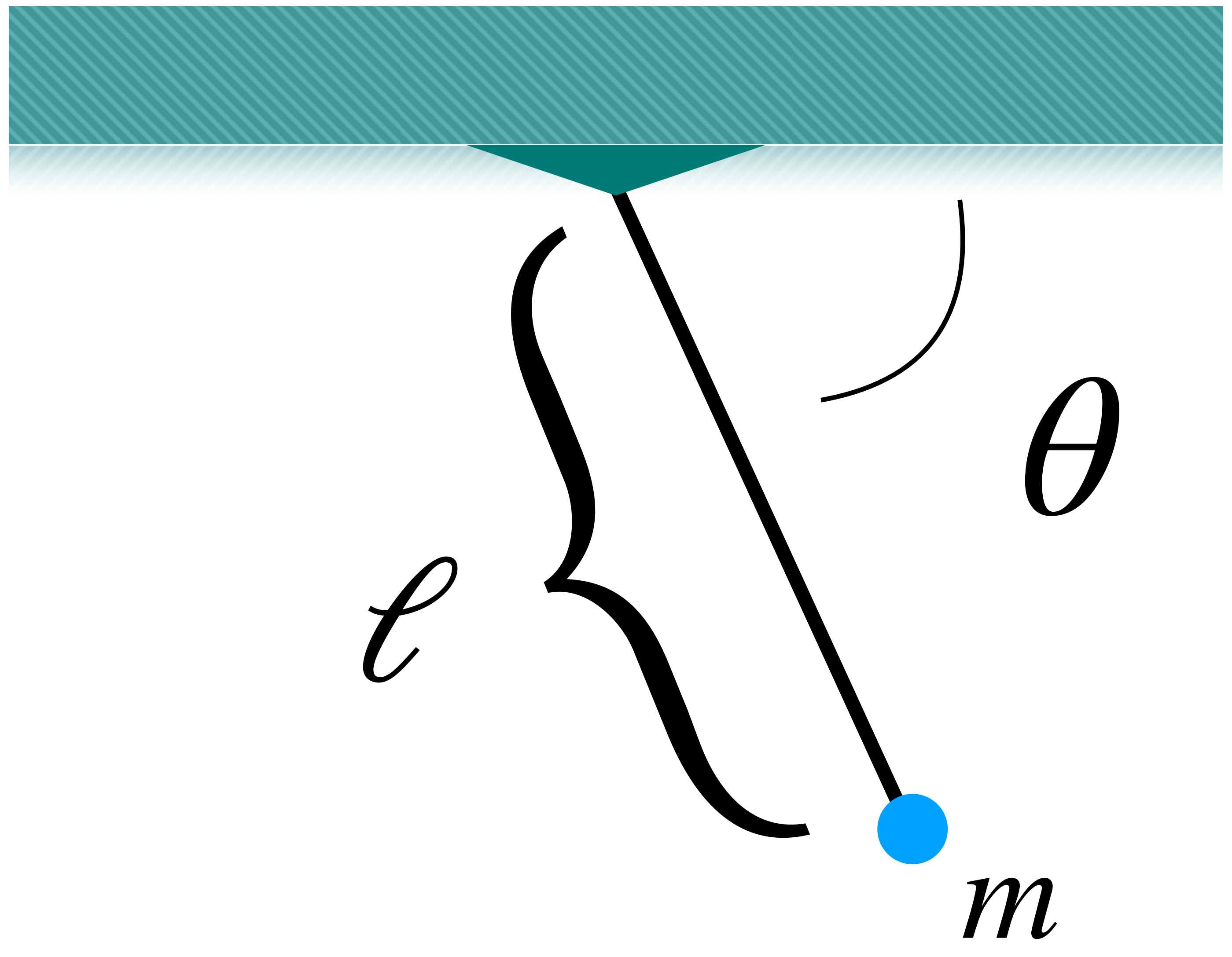
Corollary: Suppose that $f \in C^2(\mathbb{R}^n, \mathbb{R})$. If the linearized system is exponentially stable, then there exists a ball $B \subset \mathbb{R}^n$ around x^* and constants $c, \lambda > 0$ such that for every solution $x(t)$ to the nonlinear system that starts at $x(t_0) \in B$, we have

$$\|x(t) - x^*)\| \leq ce^{\lambda(t-t_0)}\|x(t_0) - x^*\|$$

- Instability is also preserved
- Marginal stability is not – consider $\dot{x} = x^3$ and $\dot{x} = -x^3$

Inverted Pendulum

$$m\ell^2\ddot{\theta} = mg\ell\sin(\theta) - b\dot{\theta} + u$$

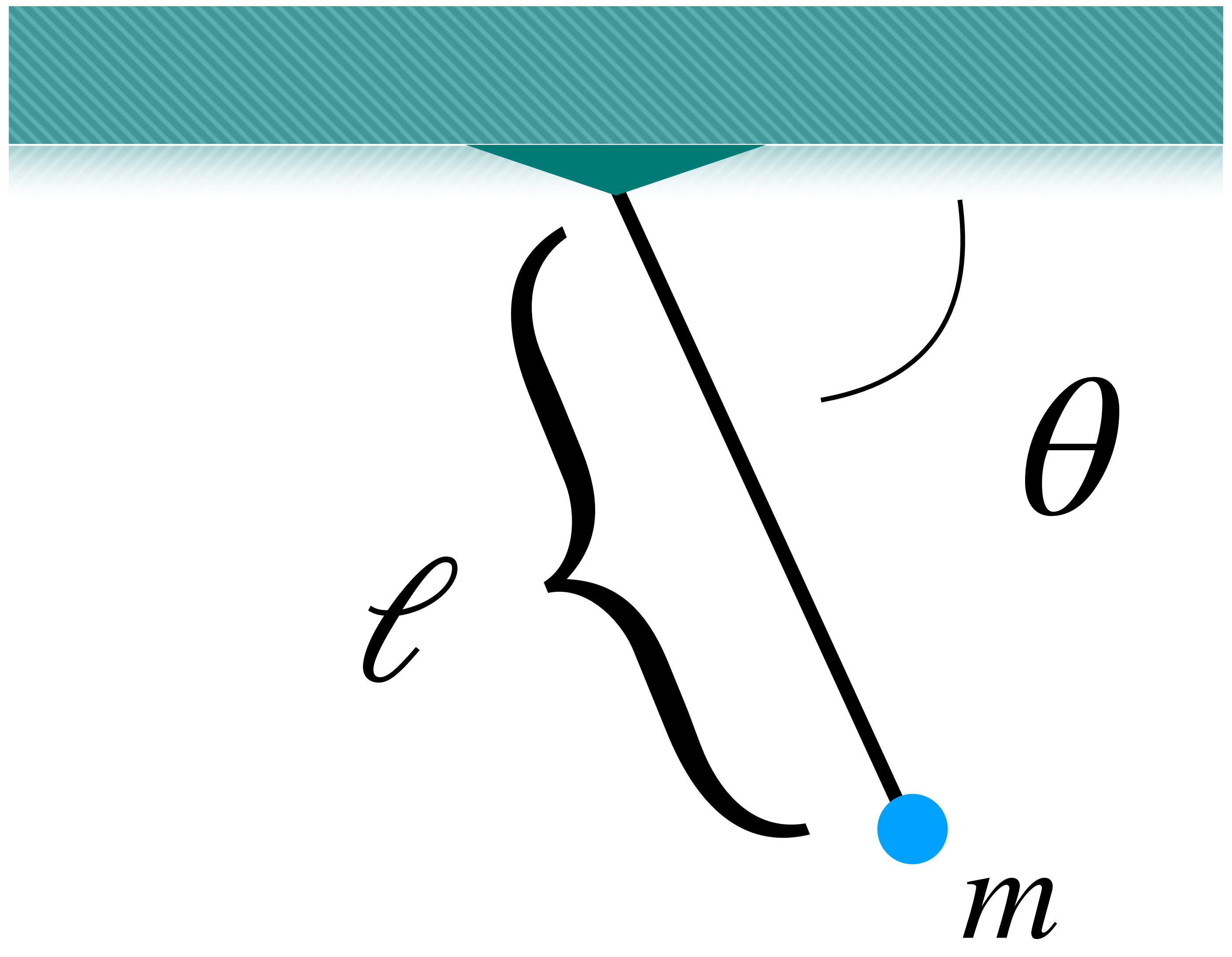


friction: b

gravity: g

State-Space Model

Inverted Pendulum: Linearized Dynamics



State-Space Model

$$\dot{x}_1 = \dot{\theta} = x_2$$

$$\dot{x}_2 = \ddot{\theta} = \frac{g}{\ell}\sin(x_1) - \frac{b}{m\ell^2}x_2 + \frac{1}{m\ell^2}\tau$$

friction: b

gravity: g

Linearization and Equilibrium:

Inverted Pendulum: Stability