AA/EE547: Wi 22

Module 3: Observability & Controllability

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References. Chapter 11 and 15 [JH]; Chapter 8/8d, [C&D]. Note that 8d in [C&D] is the discrete time chapter on controllability and observability.

Contents

Note 1: You can click on the links above under contents to bring you to the corresponding place in the pdf.

Note 2: We will primarily introduce these concepts for continuous time systems, however discrete time concepts are analogous and more detail can be found in [C&D] chapter 8d.

1 M3-RL1: Introduction to Controllability & Observability

The first component of this model seeks to introduce the basic fundamental concepts of controllability and observability of linear systems. The concepts are "dual" in some sense and hence we introduce them together.

1.1 Controllability and Rechability

Two important concepts in the analysis and synthesis of control inputs to linear systems are controllability and reachability.

Recall that if we have a linear time varying (LTV) continuous time (CT) dynamical system given by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{1}$$

then the "flow" of the ODE or state transition map is denoted $\phi(t, t_0, x_0, u)$, and we use the short hand $\phi_t(t_0, x_0, u)$ to denote the flow induced by the input u at the time t starting from initial state $x(t_0) = x_0$. The output response map is defined as $\rho(t, t_0, x_0, u)$—in particular, the output at time t starting from initial state $x(t_0) = x_0$ induced by u is

$$y(t) = \rho(t, t_0, x_0, u).$$

Let $\mathcal{U}$ denote the space of inputs where $u \in \mathcal{U}$ is a piecewise continuous function such that $u(t) \in \mathbb{R}^m$. Similarly, let $\mathcal{X}$ denote the "state-space" where $x \in \mathcal{X}$ is a piecewise continuous function such that $x(t) \in \mathbb{R}^n$. Finally, let $\mathcal{Y}$ be the space of outputs where $y \in \mathcal{Y}$ is a piecewise continuous function such that $y(t) \in \mathbb{R}^p$. With this notation, we define a dynamical system as the tuple

$$\mathcal{D} = (\mathcal{U}, \mathcal{X}, \mathcal{Y}, \phi, \rho).$$

Consider arbitrary t_0, t_1 satisfying $t_0 < t_1$, fixed initial condition $x_0 := x(t_0)$ and an input $u_{[t_0,t_1]}$ defined on the interval $[t_0, t_1]$ as indicated by the subscript notation. We say the input $u_{[t_0,t_1]}(\cdot)$ "steers" x_0 at t_0 to x_1 at t_1 if

$$x_1 := x(t_1) = \phi(t_1, t_0, x_0, u_{[t_0,t_1]}) \in \mathbb{R}^n.$$

Indeed, we know that the solution ϕ of (1) is

$$x_1 = x(t_1) = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau,$$

where $\Phi(\cdot, \cdot)$ denotes the system's state transition matrix.

Notation. Here let's take a little break to make sure notation is clear. First, elements in the spaces $\mathcal{X}$, $\mathcal{U}$ or $\mathcal{Y}$ are mappings from $\mathbb{R}_+$ to $\mathbb{R}^n$, $\mathbb{R}^m$, or $\mathbb{R}^p$, respectively. That is, $x \in \mathcal{X}$ is a mapping such that

$$x : t \in \mathbb{R}_+ \mapsto x(t) \in \mathbb{R}^n.$$

Elements in the spaces $\mathcal{U}$ and $\mathcal{Y}$ are defined similarly.

Controllability describes the condition under which inputs exist such that the system state can transferred from an arbitrary position in the state space to any other arbitrary position in the state space. The following is the definition of controllability.

Definition 1 (Controllable)**.** The system representation $\mathcal{D}$ is controllable on $[t_0, t_1]$ if for all $(x_0, x_1) \in \mathbb{R}^n$, there exists $u_{[t_0,t_1]} \in \mathcal{U}$ which steers x_0 at t_0 to x_1 at t_1.

We tend to break controllability into two different concepts:

- Controllability from the origin (reachability):
- Controllability to the origin (often simply referred to as contorllability, hence, one should take care to identify the precise definition in the given reference they are looking at).

We will provide more detail on these two concepts in the next section (**M3-RL2**).

The following proposition connects the controllability definition to surjectivity of the map $\phi(t_1, t_0, x_0, \cdot)$.

Warning! Review your [510] notes on surjectivity and injectivity of linear maps! These concepts will be used throughout this model and it is assumed you are familiar with them and the Finite Rank Operator Lemma (cf. [510] lecture notes linked here or Appendix A of **[C&D]**). The Finite Rank Operator Lemma is referred to as the "Fundamental Theorem of Linear Equations" in **[JH]** and can be found in chapter 11 of that text, specifically Theorem 11.1.

Proposition 2. The dynamical system $\mathcal{D}$ is controllable on $[t_0, t_1] \iff$ for all $x_0 \in \mathbb{R}^n$, the map

$$\phi(t_1, t_0, x_0, \cdot) : u_{[t_0,t_1]} \mapsto x(t)$$

is **surjective**, that is it maps $\mathcal{U}_{[t_0,t_1]}$ onto $\mathcal{X}$, where $\mathcal{U}_{[t_0,t_1]}$ denotes the subspace of inputs defined on the interval $[t_0, t_1]$.

[510] **Reminder**: A function $f : X \to Y$ is surjective (onto) if and only if

$$\forall y \in Y, \ \exists x \in X, \ \text{such that } y = f(x).$$

1.1.1 Memoryless Feedback and Controllability

An important type of control input is feedback control. This is where either a mapping of the state or output is used as the control input. For example, in your undergrad control class you may have seen a PID (proportional-integral-derivative) control loop where the control input is designed to have a combination of a proportional, integral and derivative mapping of the output $y(t)$.

Recall that we say a mapping is memoryless if the output it produces at a given time is dependent only on the input at that same time. Consider two memoryless maps

$$\begin{aligned} F_s &: \mathcal{X} \to \mathcal{U} && \text{(memoryless state feedback)} \\ F_o &: \mathcal{Y} \to \mathcal{U} && \text{(memoryless output feedback)} \end{aligned}$$

Applying F_s and F_o to our dynamical system $\mathcal{D} = (\mathcal{U}, \mathcal{X}, \mathcal{Y}, \phi, \rho)$, we get resulting systems $\mathcal{D}_s$ and $\mathcal{D}_o$, respectively, which are depicted in Figures 1a and 1b, respectively.

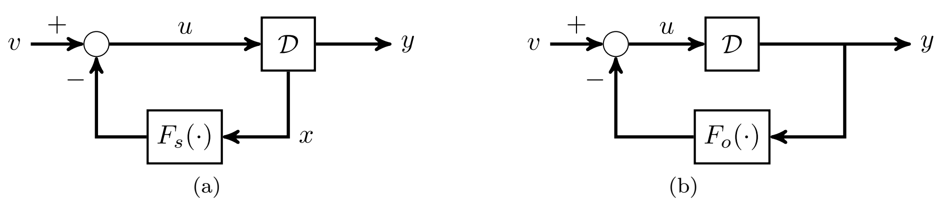


Figure 1: (a) System $\mathcal{D}_s$ with memoryless state feedback; (b) System $\mathcal{D}_o$ with memoryless state feedback.

From the above figures, we deduce

$$u(t) = v(t) - F_s(x(t)) \quad \forall\ t,$$

and

$$u(t) = v(t) - F_o(y(t)) \quad \forall\ t.$$

Assumption. (well-posedness) For $\mathcal{D}_s$ and $\mathcal{D}_o$, assume that for all (x_0, t_0), for all exogenous inputs $v(\cdot)$ there is one and only one state response $x(\cdot)$ and one output response $y(\cdot)$.

Example (of a system violating well-posedness). Consider $A = 0, B = 0, C = 0$, and $D = 1$—i.e., a single-input-single-output (SISO) linear system. Consider $u = v + y$. The closed loop system is ill-posed since $y = u - v$ and $y = u$ simultaneously. Roughly speaking, well-posedness calls for some delay around the feedback loop.

The following theorem states that controllability is preserved under memoryless state and output feedback.

Theorem 3. Let $\mathcal{D}_s, \mathcal{D}_o$ be well-posed. Then

$$\begin{aligned} &\mathcal{D} \text{ is controllable on } [t_0, t_1] \\ &\iff \mathcal{D}_s \text{ is controllable on } [t_0, t_1] \quad (1) \\ &\iff \mathcal{D}_o \text{ is controllable on } [t_0, t_1] \quad (2) \end{aligned}$$

To reiterate in a slightly different way, this theorem states that (potentially, nonlinear) memoryless state-feedback and output-feedback do not affect controllability.

Proof of Theorem 3. We provide the proof for the first equivalence, and leave the second one for the reader to show. It follows from the same style argument.

($\Longrightarrow$) By assumption $\mathcal{D}$ is controllable on $[t_0, t_1]$. Consider arbitrary (x_0, t_0) and (x_1, t_1). Since $\mathcal{D}$ is controllable on $[t_0, t_1]$, $\exists \tilde{u}_{[t_0,t_1]}(\cdot)$ steering x_0 at t_0 to x_1 at t_1 (cf. Definition 1). The state space of $\mathcal{D}$ is identical to $\mathcal{D}_s$ since F_s is memoryless. Apply to $\mathcal{D}_s$ the exogenous input defined by

$$\begin{aligned} \tilde{v}(t) &= \tilde{u}(t) + F_s(x(t)) \\ &= \tilde{u}(t) + F_s(\phi(t, t_0, x_0, \tilde{u}_{[t_0,t]})) \end{aligned}$$

Then $\tilde{v}(t)$ steers x_0 at t_0 to x_1 at t_1 by the well-posedness assumption.

($\Longleftarrow$) By controllability of $\mathcal{D}_s$, for all $x_0, x_1 \in \mathcal{X}$, $\exists\ \tilde{v}_{[t_0,t_1]}$ that steers (x_0, t_0) to (x_1, t_1) on $\mathcal{D}_s$. Since $\mathcal{D}$ and $\mathcal{D}_s$ have the same state space, by the well-posedness assumption, $\tilde{v}$ will produce a unique input $\tilde{u}$ of $\mathcal{D}$ which steers x_0 at t_0 to x_1 at t_1. □

Key Take-Away. Roughly speaking, *nonlinear memoryless* state-feedback and output-feedback do not affect controllability. The memeoryless assumption is *crucial* since it allows for us to use the *same state space* for $\mathcal{D}, \mathcal{D}_s, \mathcal{D}_o$.

1.2 Observability

The second concept we introduce is observability. This concept characterizes under what conditions we can "observe" the state of a dynamical system given the output map ρ.

Definition 4 (Observable)**.** The dynamical system $\mathcal{D}$ is called *observable* on $[t_0, t_1]$ if and only if, given, $\mathcal{D}$, for all inputs $u_{[t_0,t_1]}$ and for all corresponding outputs $y_{[t_0,t_1]} \in \mathcal{Y}$ the state x_0 at time t_0 is *uniquely* determined.

Of course, once x_0 is calculated, from $u_{[t_0,t_1]}$ and the state transition map ϕ we can calculate the state trajectory $x(\cdot)$; indeed,

$$x(t) = \phi(t, t_0, x_0, u_{[t_0,t_1]}),\ \forall\ t \in [t_0, t_1].$$

Hence, we concern ourselves with observability of the initial state x_0. Analogous to the connection between controllability and surjectivity of the state transition map (flow) ϕ, observability and injectivity are fundamentally connected.

Proposition 5. The dynamical system $\mathcal{D}$ is observable on $[t_0, t_1] \iff$ for each fixed $u_{[t_0,t_1]}$ the partial response map

$$x_0 \mapsto y_{[t_0,t_1]} = \rho(\cdot, t_0, x_0, u_{[t_0,t_1]})$$

is **injective**, that is the partial response map is a one to one map from $\mathcal{X}$ to $\mathcal{Y}$.

[510] **Reminder**: A function $f : X \to Y$ is injective (one-to-one) if and only if

$$[f(x_1) = f(x_2) \implies x_1 = x_2] \iff [x_1 \neq x_2 \implies f(x_1) \neq f(x_2)].$$

1.2.1 Memoryless Feedback and Feedforward, and observability

As with controllability, it is important to understand when observability is preserved under different types of common control input designs. We gave an example of feedback control in the preceding subsection. Another important type of control input design is known as "feedforward" control. This is where a function of the input is summed up with the output of the dynamical system $\mathcal{D}$ in order to produce some effect $y(t)$.

For a given dynamical system $\mathcal{D}$, consider the map $F_o : \mathcal{Y} \to \mathcal{U}$ and the map $F_f : \mathcal{U} \to \mathcal{Y}$ where we use F_o to apply to $\mathcal{D}$ a **memoryless output feedback** and F_f to apply to $\mathcal{D}$ a **memoryless feedforward control**. Call the resulting system $\mathcal{D}_o$ and $\mathcal{D}_f$, resp. (see Figs. 2a and 2b).

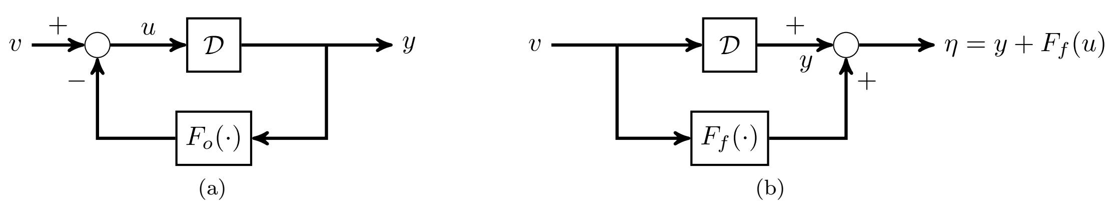


Figure 2: (a) System $\mathcal{D}_o$ with memoryless output feedback; (b) System $\mathcal{D}_f$ with memoryless feedforward.

The following theorem states that observabilility is preserved under feedforward and output feedback control.

Theorem 6. For the system $\mathcal{D}_f$ and the system $\mathcal{D}_o$ satisfying Assumption 1.1.1, we have

$$\begin{aligned} &\mathcal{D} \text{ is observable on } [t_0, t_1] \\ \iff\ &\mathcal{D}_o \text{ is observable on } [t_0, t_1] \qquad (1) \\ \iff\ &\mathcal{D}_f \text{ is observable on } [t_0, t_1] \qquad (2) \end{aligned}$$

The proof of the above theorem is omitted but it is analogous to the proof of Theorem 3.

Remark. Memoryless state feedback may affect observability. For example, for a linear time-invariant system representation $R = [A, B, C, D]$, there may exist a linear state feedback F_s such that for some states x_0 and for some inputs $u(\cdot)$, the state trajectory remains in the nullspace of C for all t. Try and construct such a system.

2 M3-RL2: Controllability of LTV Systems

Reference: **[C&D]** 8.2–8.4; **[JH]** Chapter 11, 15

Consider a LTV system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t)\end{aligned}$$

We have seen (in **Module 1**) that for an input $u(t)$ defined on the interval $[t_0, t]$, the solution is given by

$$x(t) = \phi(t, t_0, x_0, u_{[t_0,t]}) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)\ d\tau$$

We will leverage the structure of this solution in our analysis and characterization of controllability and observability. Towards that end, we need a little reminder of linear algebra ([510]) concepts including the finite rank operator lemma and the notion of a Grammian.

2.1 A Brief Review of the Finite Rank Operator Lemma

We can characterize the controllability and observability in terms of particular linear operators.[1]

This is where the **Finite Rank Operator (FRO) Lemma** and adjoint map definition will play a role. The FRO lemma is one of my favorite concepts in linear algebra by far! For more detail you can check out the following references:

- [510] lecture notes (linked here) §2.5
- **[C&D]** Appendix A.7.4
- **[JH]**, Chapter 11 (§11.3 specifically)

Let's have a brief review of [510] concepts. Consider a linear operator $A : H \to F^m$ which maps a Hilbert space and an m–dimensional Hilbert space space F^m defined by $(F^m, F, \langle\cdot,\cdot\rangle_{F^m})$ where F is either $\mathbb{R}$ or $\mathbb{C}$ and $\langle\cdot,\cdot\rangle_{F^m}$ is the inner product associated to F^m. Recall that a **Hilbert space** is a vector space equipped with an inner product operation that enables defining a distance function and perpendicularity (aka orthogonality).

Examples (Hilbert Spaces).

1. The Euclidean vector space $\mathbb{R}^3$ equipped with the usual dot product $\langle x, y\rangle = x^\top y$ where $x, y \in \mathbb{R}^3$ is a Hilbert space.
2. The space of ℓ_2–sequences equipped with the ℓ_2 inner product. A sequence $\{x_n\}$ is said to belong to ℓ_2 if
$$\sum_{n=1}^{\infty} |x_n|^2 < \infty.$$
The ℓ_2 inner product is
$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \bar{y}_n.$$
3. The space $L^2(X, \mu)$ of complex-valued measurable functions on X for which the following holds:
$$\int_X |f|^2 d\mu < \infty.$$
The inner product on this space is
$$\langle f, g\rangle = \int_X f(t)\overline{g(t)}\ d\mu(t)$$
A simple example of this is $L^2(\mathbb{R})$ with the usual Lebesgue measure μ.

[1]Context: The study of linear dynamical systems both in the time and frequency domain reduces to linear algebra in the majority of cases. This is why we put so much emphasis on students having a strong background in linear algebra and hence, the requirement of taking [510] which is designed to catch students up if they missed this background as an undergraduate.

The first two examples are finite dimensional while the third is infinite dimensional. In fact, in this class we need the infinite dimensional variant since our inputs, outputs, and states are functions and we will see that the Grammian operators we use to characterize observability and controllability are thus *functionals* defined on infinite dimensional Hilbert spaces.

The FRO Lemma tells us that for a map linear map $A : \mathcal{U} \to \mathcal{V}$, the following decompositions hold:

$$\mathcal{V} = \mathrm{Im}(A) \overset{\perp}{\oplus} \mathrm{Ker}(A^*)$$

and

$$\mathcal{U} = \mathrm{Im}(A^*) \overset{\perp}{\oplus} \mathrm{Ker}(A)$$

where $\mathrm{Im}(A)$ is the range space of A (or image) and $\mathrm{Ker}(A)$ is the null space of A (or kernel). The notation $\overset{\perp}{\oplus}$ denotes the orthogonal direct sum. The illustration in Figure 3 summarizes this decomposition.

Moreover, we have that

$$\begin{aligned} \mathrm{Ker}(AA^*) &= \mathrm{Ker}(A^*), \quad \mathrm{Im}(AA^*) = \mathrm{Im}(A) \\ \mathrm{Ker}(A^*A) &= \mathrm{Ker}(A), \quad \mathrm{Im}(A^*A) = \mathrm{Im}(A^*) \end{aligned}$$

and

$$AA^*|_{\mathrm{Im}(A)} \to \mathrm{Im}(A) \text{ and } A^*A|_{\mathrm{Im}(A^*)} \to \mathrm{Im}(A^*) \text{ are one-to-one and onto.}$$

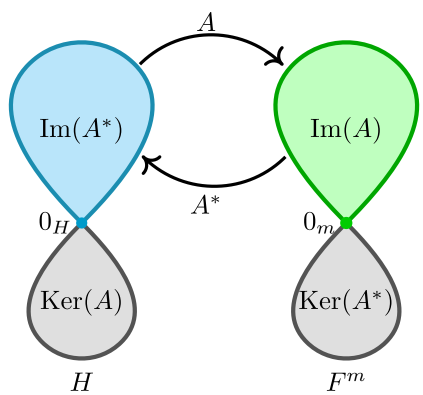


Figure 3: The orthogonal decomposition of the domain and the co-domain of a finite rank operator $A : H \to F^m$ and its associated bijections.

2.2 Controllability and Reachability of LTV

We will use the FRO lemma to aid in constructing controllability and observability grammians which are maps that can be used to characterize not only controllability and observability for a dynamical system, but can be used to construct the controllable (and reachable) and observable subspaces.

Consider the dynamical system

$$\left.\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x \end{aligned}\right\} \qquad (\mathcal{D})$$

where for simplicity, we take $D \equiv 0$. We use $(A(\cdot), B(\cdot))$ as short hand for identifying the dynamical system when referring to controllability and $(A(\cdot), C(\cdot))$ as short hand when referring to observabiltiiy. The reason for this is that we will see that these respective components of $\mathcal{D}$ are all that are need to characterize the corresponding concepts. In fact, we often simply say the pair $(A(\cdot), B(\cdot))$ (resp. $(A(\cdot), C(\cdot))$) is controllable (resp. observable).

Recall from Definition 1 that the dynamical system $\mathcal{D}$ (and hence the pair $(A(\cdot), B(\cdot))$) is controllable on $[t_0, t_1]$ if and only if $\forall(x_0, t_0)$ and $\forall(x_1, t_1)$, there exists $u(\cdot)$ that steers (x_0, t_0) to (x_1, t_1)—i.e., $x_1 := x(t_1) = \phi(t_1, t_0, x_0, u)$.

We have the following definitions for controllable to zero and controllable from zero (reachable).

Definition 7. Consider the pair $(A(\cdot), B(\cdot))$.

1. The state x_0 is controllable to zero on $[t_0, t_1]$ if and only if there exists $u_{[t_0,t_1]}$ that steers (x_0, t_0) to $(0, t_1)$.
2. The state x_1 is reachable on $[t_0, t_1]$ if and only if there exists $u_{[t_0,t_1]}$ that steers $(0, t_0)$ to (x_1, t_1).

Given the above definitions of different types of controllability, it is of interest to characterize the subspaces of $\mathcal{X}$ (and in particular $\mathbb{R}^n$) that are controllable (to zero) and reachable (controllable from the origin). We do this in particular when the entire state space is not controllable or reachable.

In fact the area of research on reachability is massive. Its highly important for understanding important concepts such as **safety**; e.g., in the communities studying hybrid systems or reinforcement learning in settings with unknown constraints and rewards across the state-action space. For the final project in this course, this is an area worth considering. There is lots of potential for it to roll over into research output.

Writing out $x(t_1)$ using the above expression for the solution, we can define the so-called reachbility map $\mathcal{L}_{r,[t_0,t_1]} : PC([t_0, t_1]) \to \mathbb{C}^n$:

$$\begin{aligned} x_1 := x(t_1) = \phi(t_1, t_0, x_0, u_{[t_0,t_1]}) &= \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau \\ &= \Phi(t_1, t_0)x_0 + \mathcal{L}_r u \end{aligned}$$

where $\mathcal{L}_{r,[t_0,t_1]}$ is defined by

$$\mathcal{L}_{r,[t_0,t_1]}(u(\cdot)) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau$$

Note we will drop the dependence on $[t_0, t_1]$ when clear from context.

The expression for x_1 shows that there will be an input $u_{[t_0,t_1]}$ that transfers an arbitrary (x_0, t_0) to an arbitrary (x_1, t_1) if and only if the map $\mathcal{L}_{r,[t_0,t_1]} : PC([t_0, t_1]) \to \mathbb{C}^n$ is **surjective** (cf Proposition 2).

Proposition 8. The following equivalence holds:

$$(A(\cdot), B(\cdot)) \text{ is controllable on } [t_0, t_1] \iff \mathcal{L}_{r,[t_0,t_1]}(u(\cdot)) \text{ is surjective.}$$

The map $\mathcal{L}_{r,[t_0,t_1]}$ determines the set of states that can be reached from the origin at some time $t = t_1$. In short, the study of the range of $\mathcal{L}_r$ is central to the study of controllability/reachability. Here, we will drop the subscript $[t_0, t_1]$ on the map $\mathcal{L}_r$ when clear from context to reduce clutter.

Definition 9 (Reachable Subspace)**.** Given the pair $(A(\cdot), B(\cdot))$, space of reachable states on the time interval $[t_0, t_1]$ is the image (or equivalently, the range) of the operator $\mathcal{L}_{r,[t_0,t_1]}$. More specifically, given two times $t_1 > t_0 \geq 0$, the reachable (or controllable from the origin) subspace $\Im(\mathcal{L}_r)$ consists of all states x_1 for which there exists an input $u : [t_0, t_1] \to \mathbb{C}^m$ that transfers the state from $x(t_0) = 0$ to $x(t_1) = x_1$—i.e.,

$$\mathrm{Im}(\mathcal{L}_r) = \left\{ x_1 \in \mathbb{C}^n :\ \exists u(\cdot),\ x_1 = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau \right\}$$

Sometimes we do not want to specify t_1 in which case we say that the pair $(A(\cdot), B(\cdot))$ is controllable (reachable) at to if and only if for some $t_1 > t_0$, the pair is controllable (reachable) on $[t_0, t_1]$.

Given that the state transition matrix $\Phi(t, t_0)$ is non-singular for all (t, t_0) (cf. **Module 1**), it is easy to prove the following result.

Theorem 10. The pair $(A(\cdot), B(\cdot))$ is completely controllable (CC) on $[t_0, t_1]$

$$\begin{aligned} &\iff \forall\, x_0 \in \mathbb{R}^n,\ \exists u_{[t_0,t_1]} \ \text{ that steers } (x_0, t_0) \text{ to } (0, t_1) \ \text{ (steering to origin)} \\ &\iff \forall x_1 \in \mathbb{R}^n,\ \exists u_{[t_0,t_1]} \ \text{ that steers } (0, t_0) \text{ to } (x_1, t_1) \ \text{ (reaching from origin)} \end{aligned}$$

Proof. Consider the interval $[t_0, t_1]$, and throughout we will use $\mathcal{L}_r$ for $\mathcal{L}_{r,[t_0,t_1]}$ for short hand. We have that

$$x_1 = \Phi(t_1, t_0)x_0 + \mathcal{L}_r u$$

Since x_0 can be zero and x_1 arbitrary, a necessary condition for CC is that $\mathrm{Im}(\mathcal{L}_r) = \mathbb{C}^n$. But this is sufficient too because if $\mathrm{Im}(\mathcal{L}_r) = \mathbb{C}^n$, given x_1 at t_1 and x_0 at t_0, $\exists\ u_{[t_0,t_1]}$ such that

$$x_1 - \Phi(t_1, t_0)x_0 = \mathcal{L}_r u$$

Both implications follow. □

2.3 Controllability in Terms of Reachability

Towards defining the analogous controllability map $\mathcal{L}_c$, we will review the computation of an adjoint (in particular, the adjoint of $\mathcal{L}_r$), and then use it to construct the reachability grammian.

2.3.1 Computing the Adjoint of the Reachability Map

Recall that the reachability map $\mathcal{L}_r : \mathcal{U}_{[t_0,t_1]} \to \mathbb{R}^n$ is defined by

$$\mathcal{L}_r(u(t_1)) = \int_{t_0}^{t_1} \Phi(t, \tau)B(\tau)u(\tau)\ d\tau \in \mathbb{R}^n.$$

Observe that $\mathcal{L}_r$ operates on the function space of inputs, which is infinite dimensional.

Claim 1. The adjoint of $\mathcal{L}_r$ is

$$\mathcal{L}_r^* x = B^*(\cdot)\Phi^*(t_1, \cdot)x$$

Proof. Since

$$\mathcal{L}_r : u_{[t_0,t_1]} \mapsto \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \in \mathbb{R}^n$$

we have that

$$\langle \mathcal{L}_r u, z\rangle_{\mathbb{R}^n} = \langle u, \mathcal{L}_r^* z\rangle_{\mathcal{U}_{[t_0,t_1]}}$$

where the inner product on the right hand side of the equality is the one associated to the Hilbert space $\mathcal{U}_{[t_0,t_1]}$. Let z^* denote the complex conjugate of the vector z. Then, the left hand side satisfies

$$\langle \mathcal{L}_r u, z\rangle_{\mathbb{R}^n} = \quad z^* \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau = \int_{t_0}^{t_1} (B^*(\tau)\Phi^*(t_1, \tau)z)^* u(\tau)d\tau = \langle u, B^*(\tau)\Phi^*(t_1, \tau)z\rangle_{\mathcal{U}_{[t_0,t_1]}}$$

so that

$$\mathcal{L}_r^* z = B^*(\cdot)\Phi^*(t_1, \cdot)z$$

as claimed. □

2.3.2 Reachability Grammian and Connections to Controllability

Given the construction of $\mathcal{L}_r^*$, we can construct the map $\mathcal{L}_r\mathcal{L}_r^* : \mathbb{R}^n \to \mathbb{R}^n$. Indeed, we have that

$$\mathcal{L}_r\mathcal{L}_r^* = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^*(\tau)\Phi^*(t_1,\tau)\ d\tau$$

This is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ and hence has a matrix representation (cf. [510] notes).

Definition 11 (Reachability Gramian.)**.** The reachability Gramian is given by the symmetric positive semi-definite matrix

$$W_{r,[t_0,t_1]} = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^*(\tau)\Phi(t_1,\tau)\ d\tau \in \mathbb{R}^{n\times n}$$

Note that by definition, $W_{r,[t_0,t_1]}$ is the integral of a semi-definite Hermitian matrix so that

$$z^*W_t(t_0,t_1)z \geq 0 \quad \forall\ z \in \mathbb{C}^n$$

It turns out that $t_1 \mapsto W_{r,[t_0,t_1]}$ solves a particular matrix differential equation—namely,

$$\dot{X}(t) = A(t)X(t) + X(t)A^*(t) + B(t)B(t)^*, \quad X(t_0) = 0 \tag{2}$$

Practice Problem. Show that $t_1 \mapsto W_t(t_0,t_1)$ solves the linear matrix differential equation (2).

Notice that critical points of the matrix differential equation (2) satisfy

$$A(t)X(t) + X(t)A^*(t) + B(t)B(t)^* = 0$$

And when the system is an LTI system (i.e., $A(t) \equiv A$ and $B(t) \equiv B$), this becomes a Lyapunov equation:

$$AX + XA^* + BB^* = 0$$

In the next section, we will see more formally the connection between controllability (and observability) grammians for LTI systems and Lypunov equations.

We can characterize (complete) controllability in terms of the reachability map and its grammian.

Theorem 12 (Controllability in terms of Reachability)**.** Let $(A(\cdot), B(\cdot))$ be given and be piecewise continuous. Then,

$$(A(\cdot),B(\cdot)) \text{ controllable on } [t_0,t_1] \iff \mathrm{Im}(\mathcal{L}_r) = \mathbb{C}^n \tag{3}$$
$$\iff \mathrm{Im}(\mathcal{L}_r\mathcal{L}_r^*) = \mathbb{C}^n \tag{4}$$
$$\iff \det(W_{r,[t_0,t_1]}) \neq 0 \tag{5}$$

Further, the set of reachable states on $[t_0,t_1]$ is the subspace $\mathrm{Im}(\mathcal{L}_r)$ which is equal to $\mathrm{Im}(W_{r,[t_0,t_1]})$.

Proof. To show the equivalence in (1), note that the left-hand side is equivalent to $\mathcal{L}_r$ being surjective which is by definition equivalent to $\mathrm{Im}(\mathcal{L}_r) = \mathbb{C}^n$.

To show the equivalence between (1) and (2), note that this is simply Finite Rank Operator (FRO) Lemma applied to $A = \mathcal{L}_r$ viewed as a map from the Hilbert space $L_2^m([t_0,t_1])$ into $\mathbb{C}^n$.[2]

To show that (2) is equivalent to (3), note that $\mathcal{L}_r\mathcal{L}_r^* : \mathbb{C}^n \to \mathbb{C}^n$, $\mathcal{L}_r\mathcal{L}_r^*$ is surjective if and only if it is a bijection (both injective and surjective) so using its matrix representation,

$$W_{r,[t_0,t_1]} = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B(\tau)^*\Phi(t_1,\tau)^*\ d\tau,$$

[2] Note that I expect you to be able to argue this without invoking FRO Lemma. That is, you should be able to construct an argument using facts about the image and kernel of the operator $\mathcal{L}_r$ and its adjoint $\mathcal{L}_r^*$.

we get that (2) is equivalent to (3).

The fact that the set of reachable states on $[t_0, t_1]$ is the subspace $\mathrm{Im}(\mathcal{L}_r)$ follows directly from the expression for $x_1 = \phi(t_1, t_0, x_0, u)$ with $x_0 = 0$. □

Remark. The essence of the reduction theorem above is that, for linear system representations, controllability on $[t_0, t_1]$, controllability to zero on $[t_0, t_1]$ of all states, and reachability on $[t_0, t_1]$ of all states are equivalent. The reader should construct a one-dimensional nonlinear example to show that this is not so for nonlinear systems.

The equivalence between controllability and reachability as described in Theorem 12 let's us define an ostensibly equivalent *controllability map*:

$$\mathcal{L}_c : u_{[t_0,t_1]} \mapsto \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\ d\tau$$

This map is derived in a similar way as the reachability map. Indeed, we say $(A(\cdot), B(\cdot))$ is controllable to zero if there exists an input $u_{[t_0,t_1]}$ that steers (x_0, t_0) to $(0, t_1)$. Writing out the solution at t_1 we have that

$$0 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau$$

Hence, we have

$$\begin{aligned}
-\Phi(t_1, t_0)x_0 &= \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau \\
\iff\ -x_0 &= \Phi(t_1, t_0)^{-1} \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau \\
&= \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\ d\tau \\
&=: \mathcal{L}_c u
\end{aligned}$$

Moreover, we can redefine the control input so that

$$0 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau \iff x_0 = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)v(\tau)\ d\tau,\ u(\cdot) = -v(\cdot)$$

Fact 13. The following equivalence holds:

$$\mathcal{L}_c \text{ is surjective} \iff \exists\ u_{[t_0,t_1]} \text{ that steers arbitrary } (x_0, t_0) \text{ to arbitrary } (x_1, t_1).$$

Note that

$$\mathrm{Im}(\mathcal{L}_r) = \Phi(t_1, t_0)\,\mathrm{Im}(\mathcal{L}_c)$$

Indeed, let $x \in \mathcal{L}_r$ then

$$x = \mathcal{L}_r u = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau$$

But, $\Phi(t_1, t_0)\Phi(t_0, \tau) = \Phi(t_1, \tau)$ so that

$$x = \mathcal{L}_r u = \Phi(t_1, t_0) \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\ d\tau$$

Definition 14 (Controllable Subspace)**.** Given two times $t_1 > t_0 \geq 0$, the controllable subspace $\mathrm{Im}(\mathcal{L}_c)$ consists of all states x_0 for which there exists an input $u : [t_0, t_1] \to \mathbb{C}^m$ that transfers the state from $x(t_0) = x_0$ to $x(t_1) = 0$—i.e.,

$$\mathrm{Im}(\mathcal{L}_c) = \{x_0 \in \mathbb{C}^n : \ \exists u(\cdot), \ 0 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau) \ d\tau\}$$

This in turn gives us an analogous theorem to the reachability theorem above.

Theorem 15. Let $(A(\cdot), B(\cdot))$ be given and be piecewise continuous. Then,

$$\begin{aligned} (A(\cdot), B(\cdot)) \text{ controllable on } [t_0, t_1] \iff & \ \mathrm{Im}(\mathcal{L}_c) = \mathbb{C}^n && (6)\\ \iff & \ \mathrm{Im}(\mathcal{L}_c\mathcal{L}_c^*) = \mathbb{C}^n && (7)\\ \iff & \ \det(W_{c,[t_0,t_1]}) \neq 0 && (8) \end{aligned}$$

where W_c is the reachability grammian

$$W_{c,[t_0,t_1]} = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B(\tau)^*\Phi(t_0, \tau)^* \ d\tau$$

Further, the set of reachable states on $[t_0, t_1]$ is the subspace $\mathrm{Im}(\mathcal{L}_c)$ which is equal to $\mathrm{Im}(W_{c,[t_0,t_1]})$.

Practice Problem. Show that given $t_1 > t_0$,

$$\mathrm{Im}(\mathcal{L}_r) = \Phi(t_1, t_0)\,\mathrm{Im}(\mathcal{L}_c)$$

and derive a matrix differential equation that $t_1 \mapsto W_{c,[t_0,t_1]}$ solves.

2.4 M3-RL2-a: Finding the Minimum Cost Control

One interesting application is to the problem of finding the minimum cost control. Consider the cost of control to be the given by the L_2-norm of $u(\cdot)$:

$$\langle u, u\rangle = \int_{t_0}^{t_1} u(t)^*u(t) \ dt = \|u\|_2^2$$

Then if $(A(\cdot), B(\cdot))$ is controllable on $[t_0, t_1]$, then for all $x_0, x_1 \in \mathbb{C}^n$, the input $\tilde{u} : [t_0, t_1] \to \mathbb{C}^m$ defined by

$$\tilde{u}(t) = B(t)^*\Phi(t_1, t)^*W_{r,[t_0,t_1]}^{-1}(x_1 - \Phi(t_1, t_0)x_0)$$

steers (x_0, t_0) to (x_1, t_1). Note this is *one* such control that gets the job done. There are potentially infinitely many others, since controllability is about 'surjectivity' of a particular linear map.

To better understand this, the condition

$$x_1 = \phi(t_1, t_0, x_0, u) = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau) \ d\tau$$

imposes n independent constraints on an infinitely dimensional space (i.e., the space where $u(\cdot)$ lies is an infinite dimensional space).

What this implies is that the set of controls that satisfy the above "constraint" forms a linear variety (or affine subspace) of codimension n. Indeed, any $u = \tilde{u} + v$ with $v \in \mathrm{Ker}(\mathcal{L}_r)$ also gets the job done.

Further, geometrically, $\tilde{u}$ is the least-cost L_2-solution iff $\|\tilde{u}\|_2$ is the minimum distance between the origin and the linear variety $\tilde{u} + \mathrm{Ker}(\mathcal{L}_r)$. Recall from last quarter that this means that $\tilde{u}$ is the least cost L_2 solution iff $\tilde{u}$ is orthogonal to the variety $\tilde{u} + \mathrm{Ker}(\mathcal{L}_r)$ which is, in turn, equivalent to $\tilde{u} \perp \mathrm{Ker}(\mathcal{L}_r)$.

Now, we get to use our friend FRO Lemma once again! By FRO Lemma, this means that

$$\tilde{u} \text{ is the least cost solution } \iff \tilde{u} \in \mathrm{Im}(\mathcal{L}_r^*) \iff \tilde{u} = \mathcal{L}_r^* \xi,\ \xi \in \mathbb{C}^n.$$

And, the minimal cost for reaching (x_1, t_1) from $(0, t_0)$ is given by

$$\|u\|_2^2 = x_1^* W_{r,[t_0,t_1]}^{-1} x_1$$

This can easily be checked by the following reasoning:

$$u_{[t_0,t_1]} \text{ transfers } (x_0, t_0) \text{ to } (x_1, t_1) \iff x_1 - \Phi(t_1, t_0)x_0 = \mathcal{L}_r u$$

Since $\tilde{u} = \mathcal{L}_r^* \xi$, we have that

$$\xi = (\mathcal{L}_r \mathcal{L}_r^*)^{-1}(x_1 - \Phi(t_1, t_0)x_0) \implies \tilde{u} = \mathcal{L}_r^* \xi = \mathcal{L}_r^* (\mathcal{L}_r \mathcal{L}_r^*)^{-1}(x_1 - \Phi(t_1, t_0)x_0)$$

Recall again from [510], that for diagonalizable positive semi-definite (PSD) Hermitian (symmetric) matrices, one can select an orthonormal eigenbasis. Hence, since $W_{r,[t_0,t_1]}$ is a PSD Hermitian matrix, we can expand it as

$$W_{r,[t_0,t_1]} = \sum_{i=1}^{n} \lambda_i v_i v_i^*$$

where (λ_i, v_i) are eigenpairs for W_r and the v_i are orthonormal. Recall that

$$\langle u, u \rangle = \int_{t_0}^{t_1} u(t)^* u(t)\ dt = \|u\|_2^2$$

(L_2 norm), so that for a unit cost $\|u\|_2 = 1$, we can reach any of the points $v_1/\sqrt{\lambda_i}$, $i = 1, \ldots, n$. Moreover, from $(0, t_0)$ we can reach any point on the ellipsoid whose semixes are $v_i/\sqrt{\lambda_i}$. Hence, if we order the eigevalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$$

then the direction v_n is the most expensive to reach and the direction v_1 is the cheapest, so that the eigenvalues of W_r measure the effectiveness of the actuators in the task of reaching states. Furthermore, thinking along these lines was the origin for reachability computation in linear systems theory.

3 M3-RL3: Observability of LTV

The observability map is constructed in a completely analogous manner to the controllability (rechability maps), hence we will spend less time focusing on the details.

Consider a linear time varying system defined by

$$\left.\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x \end{aligned}\right\} \qquad (\mathcal{D})$$

Recall that

$$y(t_1) = \rho(t_1, t_0, x_0, u_{[t_0,t_1]}) = C(t_1)\Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} C(t_1)\Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau$$

Let $\mathcal{L}_o : \mathbb{R}^n \to \mathcal{Y}_{[t_0,t_1]}$ be defined by

$$\mathcal{L}_o x_0 = C(\cdot)\Phi(\cdot, t_0)x_0$$

(that is, $\mathcal{L}_o x_0$ is an operator in $PC([t_0, t_1])$) such that

$$(\mathcal{L}_o x_0)(t) = y(t) - \int_{t_0}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)\ d\tau$$

Definition 16. The state x_0 is unobservable on $[t_0, t_1]$ if and only if its zero input response is zero on $[t_0, t_1]$.

Hence, we have the equivalence

$$x_0 \text{ is unobservable on } [t_0, t_1] \iff x_0 \in \operatorname{Ker}(\mathcal{L}_\mathrm{o})$$

Theorem 17. Given $(A(\cdot), C(\cdot))$ (piecewise continuous on $\mathbb{R}_+$), the following are equivalent:

$$\begin{aligned} (A(\cdot), C(\cdot)) \text{ is completely observable (CO) on } [t_0, t_1] &\iff \operatorname{Ker}(\mathcal{L}_\mathrm{o}) = \{0\} \\ &\iff \operatorname{Ker}(\mathcal{L}_\mathrm{o}^*\mathcal{L}_\mathrm{o}) = \{0\} \\ &\iff \det(W_{\mathrm{o},[t_0,t_1]}) \neq 0 \end{aligned}$$

where

$$W_{\mathrm{o},[t_0,t_1]} = \int_{t_0}^{t_1} \Phi(\tau, t_0)^* C(\tau)^* C(\tau) \Phi(\tau, t_0)\, d\tau$$

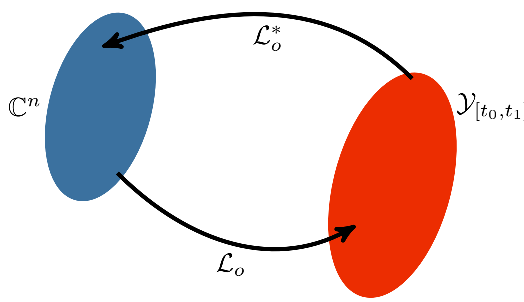


Figure 4: Graphic of Observability Map Operation

The proof of this result follows directly from the definitions of the observability map and FRO Lemma. Note: you should know how to argue the results of FRO Lemma in the context of the observability map.

Analogous to the results we had for controllability, a consequence of the above theorem is the following.

Corollary 18. Suppose that $(A(\cdot), C(\cdot))$ is observable on $[t_0, t_1]$. Then we have the following results:

a. Let y be the zero-input response due to x_0 so that

$$\langle y, y \rangle = x_0^* W_{\mathrm{o},[t_0,t_1]} x_0$$

b. Given $y_{[t_0,t_1]}$, x_0 is restricted by

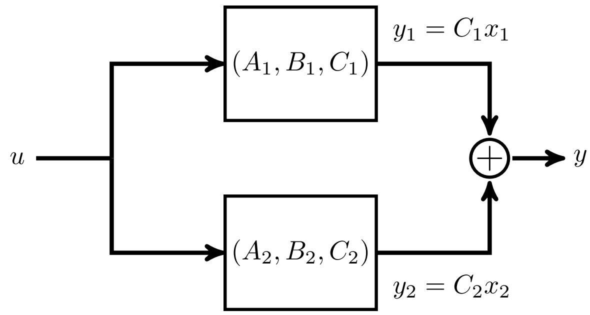
$$x_0 = (\mathcal{L}_o^*\mathcal{L}_o)^{-1}\mathcal{L}_o^* y = W_{\mathrm{o},[t_0,t_1]}^{-1} \int_{t_0}^{t_1} \Phi(\tau, t_0)^* C(\tau)^* y(\tau)\, d\tau$$

And as in the case of the controllability map, we can characterize observability in terms of the eigenstructure of W_o. Indeed, let $\lambda_n > 0$ be the smallest eigenvalue of the positive definite Hermitian matrix $W_{\mathrm{o},[t_0,t_1]}$ and en its corresponding normalized eigenvector. Then for $x_0 = e_n$, $\|x_0\|_2 = 1$ and its zero-input response is such that $\langle y, y \rangle = \lambda_n$. So, if $\lambda_n \ll 1$, some states are barely observable in case of noisy observations.

Practice Problem. Show that $t_0 \mapsto W_{\mathrm{o},[t_0,t_1]}$ is the solution to the linear matrix differential equation

$$\dot{X}(t) = -A(t)^* X(t) - X(t)A(t) - C(t)^* C(t), \; X(t_1) = 0$$

Example 19. Consider two systems connected in parallel.



The overall system has state-space model

$$\dot{x} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \quad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} x$$

The output is

$$y(t) = C_1 e^{A_1 t} x_1(0) + C_2 e^{A_2 t} x_2(t) + \int_0^t (C_1 e^{A_1(t-\tau)} B_1 + C_2 e^{A_2(t-\tau)} B_2) u(\tau) \ d\tau$$

When $A = A_1 = A_2$ and $C_1 = C_2 = C$, this reduces to

$$y(t) = C e^{At}(x_1(0) + x_2(0)) + \int_0^t C e^{A(t-\tau)}(B_1 + B_2) u(\tau) \ d\tau$$

This example demonstrates that simply knowing the input and output of the system, we cannot necessarily distinguish between initial states for which $x_1(0) + x_2(0)$ is the same value.

Theorem 20. Given t_1, t_0 with $t_1 > t_0 \geq 0$, the unobservable subspace is such that

$$\mathcal{UO}(t_0, t_1) = \mathrm{Ker}(W_{\mathrm{o},[t_0,t_1]})$$

Proof. For every $x_0 \in \mathbb{R}^n$, we have that

$$x_0^\top W_{\mathrm{o},[t_0,t_1]} x_0 = \int_{t_0}^{t_1} x_0^\top \Phi(\tau, t_0)^\top C(\tau)^\top C(\tau) \Phi(\tau, t_0) x_0 \ d\tau = \int_{t_0}^{t_1} \|C(\tau)\Phi(\tau, t_0) x_0\|^2 \ d\tau$$

so that

$$x_0 \in \mathrm{Ker}(W_{\mathrm{o},[t_0,t_1]}) \implies C(\tau)\Phi(\tau, t) x_0 = 0, \ \forall \tau \in [t_0, t_1] \implies x_0 \in \mathcal{UO}(t_0, t_1)$$

On the other hand,

$$x_0 \in \mathcal{UO}(t_0, t_1) \implies C(\tau)\Phi(\tau, t_0) x_0 = 0, \ \forall \tau \in [t_0, t_1] \implies x_0 \in \mathrm{Ker}(W_{\mathrm{o},[t_0,t_1]})$$

where we have used the fact that

$$x^\top W x = 0 \implies W x = 0$$

for W PSD. □

Remark. The key to giving constructive tests for controllability and observability is to give conditions under which $\mathrm{Im}(\mathcal{L}_c) = \mathbb{R}^n$ and $\mathrm{Ker}(\mathcal{L}_o) = \{0\}$.

4 M3-RL4: LTI Observability & Controllability

The goal today is to reduce the more complicated and abstract conditions for controllability/observability of LTV systems to the LTI case, and generate several tests for controllability and observability.

4.1 The Basics for Controllability and Observability of LTI Systems

Consider an LTI system defined by

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

with $x \in \mathbb{R}^n$.

Recall from the previous lectures in this module, that the reachability grammian is given by

$$W_{r,[t_0,t_1]} = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^*(\tau)\Phi^*(t_1,\tau)\ d\tau$$

Hence, we have in the time invariant case

$$\begin{aligned}W_r &= \int_{t_0}^{t_1} e^{A(t_1-\tau)}BB^*e^{A^*(t_1-\tau)}\ d\tau \\ &= \int_0^{t_1-t_0} e^{At}BB^*e^{A^*t}\ dt\end{aligned}$$

where we drop the index $[t_0, t_1]$ for simplicity. Similarly, the controllability grammian reduces to

$$W_c = \int_0^{t_1-t_0} e^{-At}BB^*e^{-A^*t}\ dt$$

The so-called controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{C}^{n\times nm}$$

This is a consequence of the following claim.

Claim 2. The following equality holds:

$$\mathrm{Im}(W_r) = \mathrm{Im}(\mathcal{C})$$

Proof. Consider $x_1 \in \mathrm{Im}(W_r)$ and recall that 1) $W_r = \mathcal{L}_r\mathcal{L}_r^*$ and 2) $\mathrm{Im}(\mathcal{L}_r) = \mathrm{Im}(\mathcal{L}_r\mathcal{L}_r^*)$ (by FRO Lemma). Hence, there exists an input u that transfers $x_0 = 0$ to x_1 so that

$$x_1 = \int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu(\tau)\ d\tau$$

By Cayley-Hamilton (cf. [510]), we can write

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t)A^i,\ \forall t \in \mathbb{R}$$

Thus, we have that

$$x_1 = \sum_{i=0}^{n-1} A^iB\left(\int_{t_0}^{t_1} \alpha_i(t_1-\tau)u(\tau)\ d\tau\right) = \mathcal{C}\begin{bmatrix} \int_{t_0}^{t_1}\alpha_0(t_1-\tau)u(\tau)\ d\tau \\ \vdots \\ \int_{t_0}^{t_1}\alpha_{n-1}(t_1-\tau)u(\tau)\ d\tau \end{bmatrix}$$

so that $x_1 \in \mathrm{Im}(\mathcal{C})$.

On the other hand, suppose that $x_1 \in \mathrm{Im}(\mathcal{C})$ so that $\exists v \in \mathbb{R}^{mn}$ for which $x_1 = \mathcal{C}v$. From FRO Lemma, we deduce

$$\mathrm{Im}(W_r) = \mathrm{Ker}(W_r)^\perp$$

Hence, pick an arbitrary vector $\eta_1 \in \text{Ker}(W_r)$ so that

$$\eta_1^\top e^{A(t_1-\tau)}B = 0, \ \forall \tau \in [t_0, t_1]$$

Indeed, this is easy to see by the fact that since $\eta_1 \in \text{Ker}(W_r)$,

$$\eta_1^\top W_r \eta_1 = \int_{t_0}^{t_1} \eta_1^\top e^{A(t_1-\tau)}BB^\top e^{A^\top(t_1-\tau)}\eta_1 d\tau = \int_{t_0}^{t_1} \|B^\top e^{A^\top(t_1-\tau)}\eta_1\|^2 \ d\tau = 0$$

so that

$$0 = \int_{t_0}^{t_1} u(\tau)^\top B^\top e^{A^\top(t_1-\tau)}\eta_1 d\tau$$

and, in turn, this implies that

$$B^\top e^{A^\top(t_1-\tau)}\eta_1 = 0$$

Taking k time derivatives with respect to τ, we further conclude that

$$(-1)^k \eta_1^\top A^k e^{A(t_1-\tau)}B = 0, \ \forall \tau \in [t_0, t_1], \ k \geq 0$$

and in particular for $\tau = t_1$, we obtain

$$\eta_1^\top A^k B = 0, \ \forall k \geq 0$$

It follows that $\eta_1^\top \mathcal{C} = 0$ so that

$$\eta_1^\top x_1 = \eta_1^\top \mathcal{C} v = 0, \ \forall \eta \in \text{Ker}(W_r)$$

This completes the proof of the claim. □

Analogously, the observability matrix is given by

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{C}^{np \times n}$$

4.2 Controllability Tests for LTI

We can derive "tests" based on these two matrices in order to check observability and controllability properties of the above LTI system. In the following theorem $\Delta = t_1 - t_0$ for some $t_1 > t_0$.

Theorem 21. The following are equivalent:

$$\begin{aligned} & \text{The LTI system is completely controllable on some } [0, \Delta] && (1) \\ \iff \ & \text{rank}\left(\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}\right) = n && (2) \\ \iff \ & \text{rank}\left(\begin{bmatrix} sI - A & B \end{bmatrix}\right) = n, \ \ \forall \ s \in \mathbb{C} && (3) \end{aligned}$$

Proof. [(1)$\implies$ (2)]. We know that if a system is completely controllable then the Gramian W_r is positive definite—indeed, by its construction its positive semi-definite and if it were to actually be zero at for some vector x (i.e. $x^T W_r x = 0$) then this means it drops rank which can be true if $\text{rank}(W_r) = n$:

$$\begin{aligned} W_r[t_0, t_1] &= \int_{t_0}^{t_1} e^{A(t_1-\tau)}BB^* e^{A^*(t_1-\tau)} \ d\tau \\ &= \int_0^{t_1-t_0} e^{A\tau}BB^* e^{A^*\tau} \ d\tau \\ &= \int_0^{\Delta} e^{A\tau}BB^* e^{A^*\tau} \ d\tau > 0 \end{aligned}$$

Now, suppose (2) is false. That is, $\exists\ v \in \mathbb{R}^n$ such that

$$\begin{aligned} & v^\top \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0_{nm}^\top \\ \implies & v^\top B = 0_m^\top,\ v^T AB = 0_m^\top, \dots, v^\top A^{n-1}B = 0_m^\top \\ \implies & v^\top f(A) B = 0_m^\top \text{ (by Cayley Hamilton)} \\ \implies & v^\top e^{At} B = 0_m^\top. \end{aligned}$$

Hence

$$v^\top W_r v = 0$$

which contradicts the positive definiteness of W_r.

Aside: Can you think of an alternative proof—e.g., by contrapositive?

$[(2) \implies (1)]$. Assume (2) and suppose that (1) is false. Then $\exists\ v \neq 0$ such that

$$\begin{aligned} v^T \left(\int_0^\Delta e^{A\tau} BB^* e^{A^*\tau}\ d\tau \right) v = 0 \implies & \int_0^\Delta \|B^* e^{A^*\tau} v\|^2\ d\tau = 0 \\ \implies & B^* e^{A^*\tau} v \equiv 0, \quad \forall\ \tau \in (0, \Delta) \end{aligned}$$

That is, taking derivatives, the following equalities hold:

$$\begin{aligned} B^* v &= 0, && \text{at } t = 0 \\ B^* A^* v &= 0, && \text{derivative at } t = 0 \\ &\vdots && \vdots \\ B^* (A^{n-1})^* v &= 0, && n-1\text{–th derivative at } t = 0 \end{aligned}$$

Thus, we have that

$$v^\top \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0_{nm}^\top$$

which contradicts (2).

$[(2) \implies (3)]$. Suppose (2) holds and that (3) is false. Then, $\exists\ \lambda \in \sigma(A)$ such that

$$v^\top \begin{bmatrix} \lambda I - A & B \end{bmatrix} = 0_{n+m}^\top$$

That is,

$$\lambda v^\top = v^\top A \ \text{ and } \ v^\top B = 0_m^\top$$

Hence,

$$\begin{aligned} v^\top AB &= \lambda v^\top B = 0_m^\top \\ v^\top A^2 B &= \lambda v^\top AB = 0_m^\top \\ &\vdots \\ v^\top A^{n-1} B &= \lambda v^\top A^{n-1} B = 0_m^\top \end{aligned}$$

so that

$$v^\top \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0_{nm}^\top$$

contradicting (2).

$[(3) \implies (2)]$. Suppose (3) holds and (2) does not. Consider $\mathrm{Im}(\mathcal{C})$. Since (2) does not hold, $\mathrm{Im}(\mathcal{C}) \subsetneq \mathbb{R}^n$. Note that $\mathrm{Im}(\mathcal{C})$ is an A–invariant subspace containing $\mathrm{Im}(B)$. Let V_1 be any subspace of $\mathbb{R}^n$ such that

$$\mathrm{Im}(\mathcal{C}) \oplus V_1 = \mathbb{R}^n$$

Then, by the second representation theorem, there exists a representation of A, B with respect to $\text{Im}(\mathcal{C}), V_1$ given by

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \tag{9}$$

Aside: This is actually called the controllable decomposition! It iwll be used in the next section to better understand the concept of "stabilizability".

Thus, $\exists\, T \in \mathbb{R}^{n\times n}$ such that

$$T^{-1}AT = \tilde{A} \quad \text{and} \quad T^{-1}B = \tilde{B}$$

Now

$$\text{rank}\begin{bmatrix} sI - A & B \end{bmatrix} = \text{rank}\begin{bmatrix} sI - \tilde{A} & \tilde{B} \end{bmatrix}$$

since

$$\begin{bmatrix} sI - \tilde{A} & \tilde{B} \end{bmatrix} = T^{-1}\begin{bmatrix} sI - A & B \end{bmatrix}\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}$$

Further,

$$\text{rank}\begin{bmatrix} sI - \tilde{A} & \tilde{B} \end{bmatrix} = \text{rank}\begin{bmatrix} sI - \tilde{A}_{11} & -\tilde{A}_{12} & \tilde{B}_1 \\ 0 & sI - \tilde{A}_{22} & 0 \end{bmatrix}$$

But this has rank less than n for all $s \in \sigma(\tilde{A}_{22})$, contradicting (3). □

Typically (meaning in most references), the test

$$\text{rank}(\mathcal{C}) = 0$$

is called the **controllability test**, while the test

$$\text{rank}([sI - A \;\; B]) = n, \qquad \forall\, s \in \mathbb{C},$$

is called the **PBH test for controllability** where "PBH" is an abbreviation for Popov-Belevitch-Hautus, the three namesakes of the result.

Example 22. The equations of motion of a satellite, linearized around a steady-state solution, are given by $\dot{x} = Ax + Bu$, where x_1 and x_2 denote the perturbations in the radius and the radial velocity, respectively, x_3 and x_4 denote the perturbations in the angle and the angular velocity, and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The input is a radial thruster u_1 combined with a tangential thruster u_2.

a. *is the system controllable?* Yes, this is easy to check by just computing

$$\text{rank}(\mathcal{C}) = \text{rank}\left(\begin{bmatrix} B & AB \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 \end{bmatrix}\right) = 4$$

That is we did not need to consider higher powers of A.

b. *What is the thrusters individually fail?*

If u_2 fails we have

$$\text{rank}(\mathcal{C}) = \text{rank}\left(\begin{bmatrix} B_2 & AB_2 & A^2B_2 \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & -4 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \end{bmatrix}\right) = 3$$

so its not controllable.

If u_1 fails we have

$$\operatorname{rank}(\mathcal{C}) = \operatorname{rank}\left(\begin{bmatrix} B_2 & AB_2 & A^2B_2 \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & -3 \\ 1 & 1 & -3 & -7 \end{bmatrix}\right) = 4$$

so that the system remains controllable.

Fact. Some facts:

1. Since the controllability matrix does not depend on time, if the LTI system is CC for *some* $\Delta > 0$ then it is CC for *all* $\Delta > 0$. Because of this fact, we often say that the pair (A, B) is controllable.
2. Controllability test can be done by just examining A and B without computing the grammian. The matrix-rank test is attractive in that it enumerates the vectors in the controllability subspace. However, numerically, since it involves powers of A, numerical stability needs to be considered.
3. The PBH test involves simply checking the condition at the eigenvalues. It is because for $(sI - A, B)$ to have rank less than n, s must be an eigenvalue.
4. The range space of the controllability matrix is of special interests. It is called the controllable subspace and is the set of all states that can be reached from zero-initial condition. This is A–invariant.
5. Using the basis for the controllable subspace as part of the basis for $\mathbb{R}_n$, the controllability property can be easily seen in the transformed representation in (9).

4.3 LTI Observability

The dual of the controllability theorem for LTI gives a similar theorem for observability.

Theorem 23 (LTI Observability Tests)**.** The following are equivalent:

$$\text{The LTI system is completely observable on some } [0, \Delta] \tag{1}$$

$$\iff \quad \operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n \tag{2}$$

$$\iff \quad \operatorname{rank}\left(\begin{bmatrix} sI - A \\ C \end{bmatrix}\right) = n, \quad \forall\, s \in \mathbb{C} \tag{3}$$

The proof is very similar to the one for controllability (see **[C& D]**, Chapter 8 for details and alternate proof for controllability using observability proof). This being said, it is perhaps instructive to see the proof sketch for the equivalence of (2) with (3).

Proof. Instead of considering the range space of the controllability matrix, we consider the null space (kernel) of the observability matrix. Its kernel is also A-invariant. Hence if the observability matrix is not full rank, then using basis for its kernel as the last k basis vectors of $\mathbb{R}^n$, the system can be represented as

$$\begin{aligned} \dot{z} &= \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} z + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u \\ y &= \begin{bmatrix} \tilde{C} & 0 \end{bmatrix} z \end{aligned} \tag{10}$$

where

$$C = \begin{bmatrix} \tilde{C} & 0 \end{bmatrix} T^{-1}, \quad \text{and} \quad A = T \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} T$$

and the dim of $\tilde{A}_{22}$ is non-zero (and is the dimension of the kernel of the observability matrix).

Aside: The system in (10) is actually called the observable decomposition! It iwll be used in the next section to better understand the concept of "detectability". □

Fact. Some facts:

1. Observability of a LTI system does not depend on the time interval. So, theoretically speaking, if observing the output and input for an arbitrary amount of time will be sufficient to figure out x_0. In reality, when more data is available, one can do more averaging to eliminate effects of noise (e.g. using the Least squares Kalman Filter approach).
2. The subspace of particular interest is the null space of the controllability matrix. An initial state lying in this set will generate identically 0 zero-input response. This subspace is called the unobservable subspace.
3. Using the basis of the unobservable subspace as part of the basis of $\mathbb{R}_n$, the observability property can be easily seen.

Remark. An easy extension to the proofs of the above theorems is that

$$\begin{array}{rcccl} \mathrm{Im}(W_{c,[0,\Delta]}) & = & \mathrm{Im}(\mathcal{L}_c) & = & \mathrm{Im}(\mathcal{C}) \subset \mathbb{R}^n \\ \mathrm{Ker}(W_{o,[0,\Delta]}) & = & \mathrm{Ker}(\mathcal{L}_o) & = & \mathrm{Ker}(\mathcal{O}) \subset \mathbb{R}^n \end{array}$$

4.4 Lyapunov Tests for Controllability/Observability

As noted, all the concepts we learn in this class have a strong connection with the Lyapunov equation.

Recall for LTV systems the following facts.

Fact 24. The function $t_1 \mapsto W_{r,[t_0,t_1]}$ is the solution of the linear matrix equation

$$\dot{X}(t) = A(t)X(t) + X(t)A^*(t) + B(t)B^*(t)$$

with $X(t_0) = 0$. Similarly, The map $t_0 \mapsto W_{o,[t_0,t_1]}$ is the solution to the linear matrix equation

$$\dot{X}(t) = -A^*(t)X(t) - X(t)A(t) - C^*(t)C(t)$$

with $X(t_1) = 0$.

For LTI, this gives rise to the Lyapunov tests for controllability (resp. observability). The utility of this is in the synthesis of feedback controllers that stablilize the system. You will have a homework on this.

Consider the continuous time system

$$\dot{x} = Ax + Bu, \tag{CT}$$

and analogous discrete time (DT) system,

$$x^+ = Ax + Bu \tag{DT}$$

Proposition 25. Assume that A is Hurwitz—i.e. $\mathrm{spec}(A) \subset \mathbb{C}_-^\circ$—in the continuous time case, or A is a Schur matrix—i.e., $\mathrm{spec}(A) \subset \mathbb{D}_1$—in the discrete time case. The LTI system is controllable if and only if there is a unique positive-definite solution W to the following Lyapunov equation

$$AW + WA^\top = -BB^\top,$$

Moreover, the unique solution is

$$W = \int_0^\infty e^{A\tau} BB^\top e^{A^\top \tau} \, d\tau$$

Analogously, in the discrete time case, there is a unique positive definite solution W to the following Lyapunov (Schur) equation

$$AWA^\top - W = -BB^\top,$$

and it is expressed as

$$W = \sum_{t=0}^{\infty} A^t BB^\top (A^\top)^t$$

4.5 Stabilizing via Feedback

One of the important applications of controllability is that if we know our system is controllable then we can design a feedback controller that stabilizes the system. Let χ_A denote the characteristic polynomial of A—i.e,

$$\chi_A = \det(sI - A).$$

Proposition 26. For matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and for any monoic real polynomial π of degree n, there exists $F \in \mathbb{R}^{m\times n}$ such that

$$\chi_{A+BF} = \pi$$

if and only if the pair (A, B) is controllable.

The interpretation of the above proposition is through the idea of *constant state feedback*. Suppose that the state variables are available, (from, say, measurements). Then, we calculate Fx for a given $F \in \mathbb{R}^{m\times n}$ and we feedback Fx to the input: the resulting feedback system is, with $u = 0$,

$$\dot{x} = (A + BF)x$$

Thus the proposition asserts that the pair (A, B) is controllable if and only if we can always choose F so that the closed-loop characteristic polynomial χ_{A+BF} has as roots a list of n preassigned points in $\mathcal{C}$; of course, these n points must be located symmetrically with respect to the real axis because the polynomial χ_{A+BF} has real coefficients.

Key Take Away: This is to say that given any unstable A with (A, B) controllable we can always stabilize it by constant state feedback.

4.6 Controllable Canonical Form

Consider a single-input-single-output LTI system that is completely controllable. We claim that there exists a similarity transformation that converts the system to the form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

Indeed, since the system is completely controllable, the controllability matrix $\mathcal{C}$ has rank n and is invertible so that

$$\mathcal{C}^{-1} = \begin{bmatrix} \tilde{\mathcal{C}} \\ q \end{bmatrix}$$

where q is the last row of the matrix inverse. That is,

$$\mathcal{C}^{-1} = \begin{bmatrix} \tilde{\mathcal{C}} \\ q \end{bmatrix} \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} = I$$

so that

$$qA^{i-1}b = \begin{cases} 0, & i = 1, \dots, n-1 \\ 1, & i = n \end{cases}$$

Then

$$\begin{bmatrix} qb \\ qAb \\ \vdots \\ qA^{n-1}b \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \bar{B}$$

and by Cayley-Hamilton we have that

$$\begin{aligned} \begin{bmatrix} qA \\ qA^2 \\ \vdots \\ qA^{n-1} \\ qA^n \end{bmatrix} &= \begin{bmatrix} qA \\ \vdots \\ qA^{n-1} \\ -q\sum_{i=1}^{n} a_{n-i+1}A^{i-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} q \\ qA \\ \vdots \\ qA^{n-2} \\ qA^{n-1} \end{bmatrix} \\ &= \bar{A}T \end{aligned}$$

where

$$T = \begin{bmatrix} q \\ qA \\ \vdots \\ qA^{n-1} \end{bmatrix}$$

Hence, $A = T^{-1}\bar{A}T$ and $b = T^{-1}\bar{B}$.

Note. Any system that can be placed in controllable canonical form can be stabilized by state feedback.

Example 27. Consider

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

With state feedback the input is $u = -Kx$ where K is a constant row vector. Consider the polynomial $p(a) = \sum_{k=0}^{3} a_k s^{3-k} = a_0 s^3 + a_1 s^2 + a_2 s + a_3$ with $a_0 = 1$. For the closed loop feedback system with $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$, we have

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 - k_1 & -\alpha_2 - k_2 & -\alpha_1 - k_3 \end{bmatrix}$$

Then, $\chi(s) = s^3 + (\alpha_1 + k_3)s^2 + (\alpha_2 + k_2)s + \alpha_3 + k_1$. Equating the coefficients with $p(s)$ we get $a_1 = \alpha_1 + k_3$, $a_2 = \alpha_2 + k_2$, and $a_3 = \alpha_3 + k_1$. So, $K = [a_3 - \alpha_3 \;\; a_2 - \alpha_2 \;\; a_1 - \alpha_1]^T$.

Example 28. Consider

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

and the desired characteristic polynomial $p(s) = (s+1)(s+3)$. First,

$$\operatorname{rank} \mathcal{C} = \operatorname{rank}\left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right) = 2$$

Then for $u = -kx$,

$$\begin{aligned}
\det(sI - A + bk) &= \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} s - 1 + k_1 & k_2 \\ k_1 & s - 2 + k_2 \end{bmatrix}\right) \\
&= (s - 1 + k_1)(s - 2 + k_2) - k_2 k_1 \\
&= (s-1)(s-2) + k_1(s-2) + k_2(s-1) \\
&= s^2 - 3s + 2 + k_1 s - 2k_1 + k_2 s - k_2 \\
&= s^2 + (k_1 + k_2 - 3)s + 2 - 2k_1 - k_2
\end{aligned}$$

So then by equating coefficients of the above and

$$p(s) = s^2 + 4s + 3$$

we get

$$\begin{array}{rcl}
4 & = & k_1 + k_2 - 3 \implies 7 - k_2 = k_1 \\
3 & = & 2 - 2k_1 - k_2 \implies 1 = -2k_1 - k_2 \implies 1 = -2(7 - k_2) - k_2 = -14 + k_2
\end{array}$$

so that

$$k_1 = -8 \text{ and } k_2 = 15$$

and the closed loop system is thus

$$\dot{x} = (A - BK)x = \begin{bmatrix} 9 & 15 \\ 8 & 17 \end{bmatrix} x$$

5 M3-RL5: Stabilizability and Detectability

It is often useful to characterize when a system that is not fully controllable can be stabilized, and analogously when a system that is fully observable can still be detectable. To this end, we first start with decomposing the state space.

5.1 Extracting unobservable/uncontrollable dynamics

We have the following result stating that the kernel of the observability matrix and the image of the controllability matrix are A-invariant.[3]

Proposition 29. Consider the LTI system (A, B, C).

a. The set of all unobservable states is the A-invariant subspace $\operatorname{Ker}(\mathcal{O}) \subset \mathbb{C}^n$
b. The controllable subspace is the A-invariant space $\operatorname{Im}(\mathcal{C})$

[3] The notion of A-invariant subspaces is covered in **[510]**. As a reminder, a space U is A-invariant if and only if for any $u \in U$, we have that $Au \in U$.

Proof. We prove only the first claim. Suppose that $v \in \mathrm{Ker}(\mathcal{O})$ so that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ \vdots 0 \end{bmatrix}$$

We claim that $Av \in \mathrm{Ker}(\mathcal{O})$. Indeed,

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} Av = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix} v \tag{11}$$

From the fact that $v \in \mathrm{Ker}(\mathcal{O})$ we have that $CA^k v = 0$ for all $k \in \{0, \ldots, n-1\}$ so that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} Av = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{n-1}v \\ CA^n \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ CA^n v \end{bmatrix} \tag{12}$$

Then, by the Cayley Hamilton theorem which states that any matrix A satisfies its only characteristic polynomial—i.e., $\chi_A(A) = 0$—(cf. [510]), we have that

$$A^n = -c_{n-1}A^{n-1} - \cdots - c_1 A - c_0 I$$

so that

$$CA^n = (-c_{n-1}CA^{n-1} - \cdots - c_1 CA - c_0 C)v$$

and we know from the fact that $v \in \mathrm{Ker}(\mathcal{O})$ that all these terms are zero. □

Q: Can you prove the second claim?

Unobservable states. Suppose, for simplicity, that $D = 0$ and that the LTI system given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{13}$$

is unobservable. Since the system is unobservable we have that $\dim \mathrm{Ker}(\mathcal{O}) =: r < n$, the unobservable subspace is r–dimensional. Choose a basis for $\mathrm{Ker}(\mathcal{O})$ and precede these r basis-vectors with $n - r$ vectors from $\mathbb{C}^n$ so that we have a new basis for $\mathbb{C}^n$ (possible by the basis completion theorem [510]). Create a matrix T from these vectors.

Proposition 29 gives us that $\mathrm{Ker}(\mathcal{O})$ is A–invariant. By the definition of the observability matrix, we have that $\mathrm{Ker}(C) \supset \mathrm{Ker}(\mathcal{O})$ which implies that the last r basis vectors are in the nullspace of C.

Thus, in the new basis, the system is represented by

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} A_{\mathrm{o}} & 0 \\ A_{21} & A_{\mathrm{uo}} \end{bmatrix}}_{T^{-1}AT} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} B_{\mathrm{o}} \\ 0 \end{bmatrix}}_{T^{-1}B} u \\ y &= \underbrace{\begin{bmatrix} C_{\mathrm{o}} & 0 \end{bmatrix}}_{CT} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

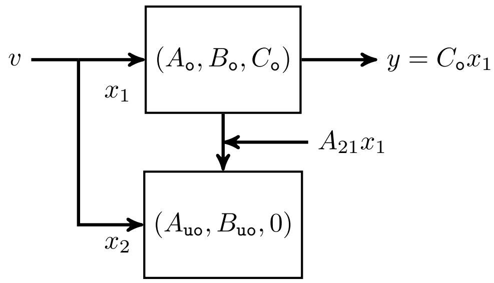


Figure 5: Extracting the unobservable part.

where $(x_1, x_2) \in \mathbb{C}^{n-r} \times \mathbb{C}^r$. The above equations imply that the system looks like Fig. 5.

As the figure shows, we have extracted an r–dimensional subsystem from R and the state x_2 of that subsystem is unobservable. Since by Proposition 29.a the set of all unobservable states is of dimension r the pair $(C_{\mathsf{o}}, A_{\mathsf{c}})$ is observable.

Controllable Part. By Proposition 29 we have that the image of the controllability matrix is A-invariant. Consider a system that is not completely controllable. Let $\dim \mathrm{Im}(\mathcal{C}) =: q < n$—that is, the controllable subspace is q–dimensional. Choose a basis for $\mathrm{Im}(\mathcal{C})$ and complete it $n-q$ vectors from $\mathbb{C}^n$ to obtain a basis for $\mathbb{C}^n$. As with the observable decomposition, let T be the similarity transform defined by stacking up these basis vectors.

By the definition of the controllability matrix $\mathrm{Im}(\mathcal{C}) \supset \mathrm{Im}(B)$, hence $\mathrm{Im}(B)$ is in the subspace generated by the first q basis vectors. Consequently, in this new basis, the system representation is of the form

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{\mathsf{c}} & A_{12} \\ 0 & A_{\mathsf{uc}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B_{\mathsf{c}} u \\ y &= \begin{bmatrix} C_{\mathsf{c}} & C_{\mathsf{uc}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

where $(x_1, x_2) \in \mathcal{C}^q \times \mathbb{C}^{n-q}$. As Figure 6 shows, we have extracted an $(n-q)$–dimensional subsystem that is totally unaffected by the input u, hence is uncontrollable. The set of controllable states is q-dimensional, $x_1 \in \mathbb{C}^q$ and all the states of the second $(n-q)$–dimensional subsystem are unaffected by u, the pair $(A_{\mathsf{c}}, B_{\mathsf{c}})$ is controllable.

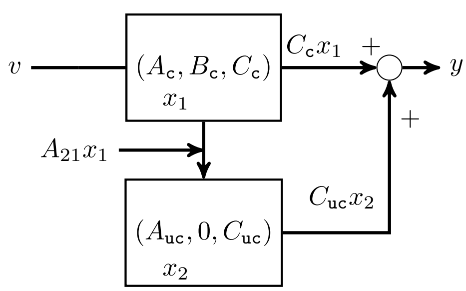


Figure 6: Extracting the unobservable part.

5.2 Kalman Decomposition

The entire state space and system can be decomposed into the parts that are observable-controllable, unobservable-controllable, observable-uncontrollable and finally unobservable-uncontrollabe.

Let $\text{Im}(\mathcal{C})$ be the reachable or controllable subspace.

Proposition 30. The following hold:

1. $\text{Im}(\mathcal{C})$ is A-invariant.
2. $\text{Im}(\mathcal{C}) = \text{Im}(B) + \text{Im}(AB) + \cdots + \text{Im}(A^{n-1}B)$ (sum of subspaces not direct sum)

Let $\text{Ker}(\mathcal{O})$ be the unobservable subspace.

Proposition 31. The following hold:

1. $\text{Ker}(\mathcal{O})$ is A-invariant.
2. $\text{Ker}(\mathcal{O}) = \text{Ker}(C) \cap \text{Ker}(CA) \cap \cdots \cap \text{Ker}(CA^{n-1})$

In addition, we have the following result.

Proposition 32. $\text{Im}(\mathcal{C})$ is the smallest A-invariant subspace containing $\text{Im}(B)$ and $\text{Ker}(\mathcal{O})$ is the largest A-invariant subspace contained in $\text{Ker}(C)$.

With these reminders we can now generate the Kalman decomposition. Let

$$\mathbb{R}^n = \text{Im}(\mathcal{C}) \oplus V_1 \quad \text{and} \quad \mathbb{R}^n = V_2 \oplus \text{Ker}(\mathcal{O})$$

where V_1 and V_2 are (any) direct summands of $\text{Im}(\mathcal{C})$ and $\text{Ker}(\mathcal{O})$, respectively.

Define

$$\Sigma_{co} = \text{Im}(\mathcal{C}) \cap V_2,\ \Sigma_{\not c o} = V_1 \cap V_2,\ \Sigma_{c \not o} = \text{Im}(\mathcal{C}) \cap \text{Ker}(\mathcal{O}),\ \Sigma_{\not c \not o} = V_1 \cap \text{Ker}(\mathcal{O})$$

Clearly

$$\text{Im}(\mathcal{C}) = \Sigma_{co} \oplus \Sigma_{c\not o},\ \text{Ker}(\mathcal{O}) = \Sigma_{\not c o} \oplus \Sigma_{\not c \not o}$$

and

$$V_1 = \Sigma_{\not c o} \oplus \Sigma_{\not c \not o},\ V_2 = \Sigma_{co} \oplus \Sigma_{\not c o}$$

and

$$\mathbb{R}^n = \Sigma_{co} \oplus \Sigma_{c \not o} \oplus \Sigma_{\not c o} \oplus \Sigma_{\not c \not o}$$

Applying these similarity transforms gives us a representation of A, B and C as

$$\begin{aligned}\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} &= \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix}\end{aligned} \tag{14}$$

More detail can be found in Chapter 16 of **[JH]** or Chapter 8.6 of **[C& D]**. Figure 7 depicts the block diagram for this decomposition.

The main result is the following theorem.

Theorem 33. For every LTI system (A, B, C) there is a similarity transformation that takes it to the form (14) for which

1. The pair

$$\left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right)$$

   is controllable

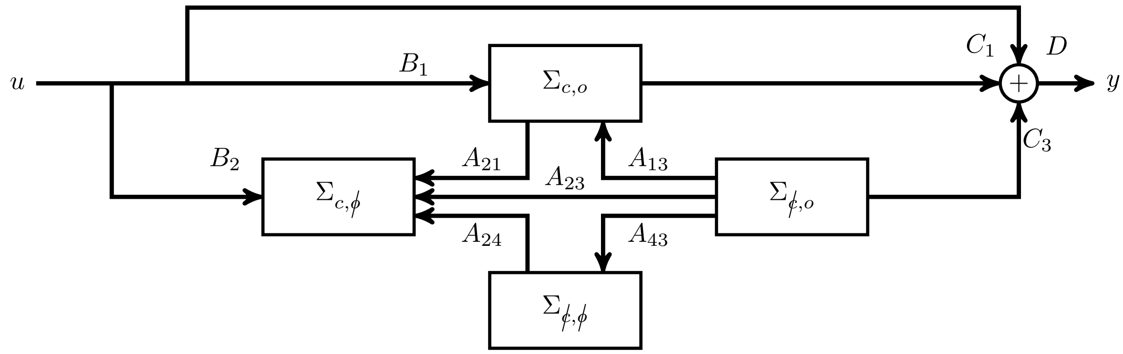


Figure 7: Kalman Decomposition

2. The pair
$$\left(\begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix}, \begin{bmatrix} C_1 & C_3 \end{bmatrix}\right)$$
is observable.
3. The triple (A_{11}, B_1, C_1) is controllable and observable.

5.3 Stabilizability and Detectability

With these fundamental decompositions (by way of similarity transformations) to the different subspaces, we can discuss stabilizability and detectability.

5.3.1 Stabilizability

We saw in Section 5.1 (and the proof of the controllability theorem for LTI) that any LTI system is "similar" (again by way of a similarity transform) to the standard form for uncontrollable systems:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{\mathtt{c}} & A_{12} \\ 0 & A_{\mathtt{uc}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B_{\mathtt{c}} u \\ y &= \begin{bmatrix} C_{\mathtt{c}} & C_{\mathtt{uc}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{15}$$

where

- we denote the controllable states as x_1 and uncontrollable states as x_2;
- the matrices $A_{\mathtt{c}}$ and $A_{\mathtt{uc}}$ correspond to the dynamics of the controllable and uncontrollable states, respectively;
- the matrix $B_{\mathtt{c}}$ corresponds to the coefficient of the control input to the controllable states;
- and, finally, the matrices $C_{\mathtt{uc}}$ and $C_{\mathtt{c}}$ correspond to the transformed output matrix C to these new coordinates.

The details on how to construct this representation follow from results in **[510]**, and is described in §5.1.

Definition 34. The pair (A, B) is stabilizable if there is a similarity transform to the form (15) with $A_{\mathtt{u}}$ Hurwitz stable.

Stabilizability can be viewed as an infinite-time version of controllability in the sense that if a system is stabilizable, then its state can be transferred to the origin from any initial state, but this may require infinite time.

Theorem 35. The following are equivalent:

1. The continuous-time LTI system (A, B) is stabilizable
2. Every eigenvector of $A^\top$ corresponding to an eigenvalue with a positive or zero real part is not in the kernel of $B^\top$.
3. (PBH test) $\text{rank}([A - \lambda I \;\; B]) = n$ for all $\lambda \in \mathbb{C}$ such that $\text{Re}(\lambda) \geq 0$.
4. There is a positive definite solution $P = P^\top \succ 0$ to the Lyapunov matrix inequality

$$AP + PA^\top - BB^\top < 0$$

Like with controllability we can leverage the Lyapunov test for stabilizability in item 3 above to synthesize stabilizing feedback controllers.

Controller Synthesis. Consider

$$\dot{x} = Ax + Bu$$

and suppose this system is stabilizable (i.e. all unstable modes are in the controllable subspace). Let $K := \frac{1}{2}B^\top P^{-1}$ where $P = P^\top \succ 0$ solve the Lyapunov matrix inequality

$$AP + PA^\top - BB^\top < 0$$

This inequality can be rewritten as

$$(A - \frac{1}{2}BB^\top P^{-1})P + P(A - \frac{1}{2}BB^\top P^{-1})^\top = (A - BK)P + P(A - BK)^\top < 0$$

Multiplying this equation on both sides by $Q := P^{-1}$, we obtain

$$Q(A - BK) + (A - BK)^\top Q < 0$$

so that since $Q \succ 0$, by the Lyapunov stability theorem $A - BK$ is Hurwitz stable. This in turn means that the controller $u = -Kx$ asymptotically stabilizes the system (A, B).

5.3.2 Detectability

As with the standard form for uncontrollable systems, there is a standard form for observable systems obtainable by way of a similarity transform. The unobservable form is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_\mathsf{o} & 0 \\ A_{21} & A_\mathsf{uo} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_\mathsf{o} \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} C_\mathsf{o} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{16}$$

as shown in §5.1.

Definition 36. A pair (A, C) is detectable if it is similar to a system in the standard form (16) with A_uo a Hurwitz matrix.

The above definition is stating that all unobservable modes are stable.

Theorem 37 (Detectability Tests)**.** The following are equivalent:

1. The continuous-time LTI system (A, C) is detectable
2. Every eigenvector of A corresponding to an eigenvalue with a positive or zero real part is not in the kernel of C.

3. (PBH test)
$$\operatorname{rank}\left(\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}\right) = n, \quad \forall \lambda \in \mathbb{C}: \ \operatorname{Re}(\lambda) \geq 0.$$

4. There is a positive definite solution $P = P^\top \succ 0$ to the Lyapunov matrix inequality
$$AP + PA^\top - C^\top C < 0$$

Observer Synthesis. Analogous to the synthesis of stabilizing feedback, we can also use the tools in this module to synthesis observers. This amounts to designing a state estimation scheme. Consider the continuous time system
$$\dot{x} = Ax + Bu, \ y = Cx + Du$$
and let $u = -Kx$ be a stabilizing feedback controller. When only the output y can be measured, the control law cannot be implemented, but if the pair (C, A) is detectable, it should be possible to estimate x from the system's output up to an error that vanishes as $t \to \infty$.

We have already seen that for an observable system, the state can be recovered from the input and output over an interval $[t_0, t_1]$ using the observability Grammian. This just gives the value at a particular time. What we want to do is design a method of recovering the state for all time.

The typical state feedback topology is depicted in Figure 8.

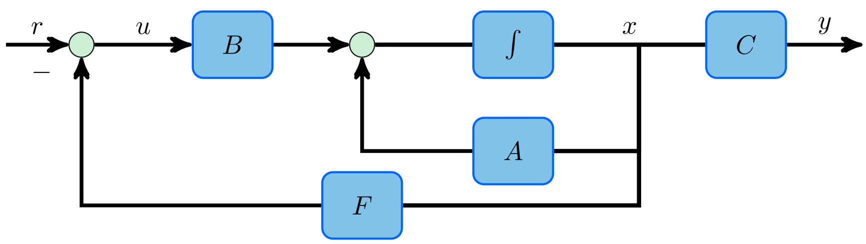


Figure 8: State Feedback Topology/Block Diagram

However, often the state vector is inaccessible for direct measurement. Techniques exist to estimate the state.

An *observer* is a signal reconstruction device which provides an estimate of inaccessible (aka unobservable) states.

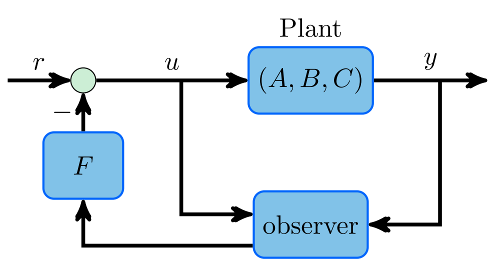


Figure 9: Observer/Plant Block Diagram

There are several ways to derive the state equations for the full-state observer. The approach in these notes is to model the observer state equations as a model of the actual system plus a correction term based on the measured output and the estimate of what that output is expected to be.

With the actual system described by
$$\dot{x} = Ax + Bu, \ y = Cx$$

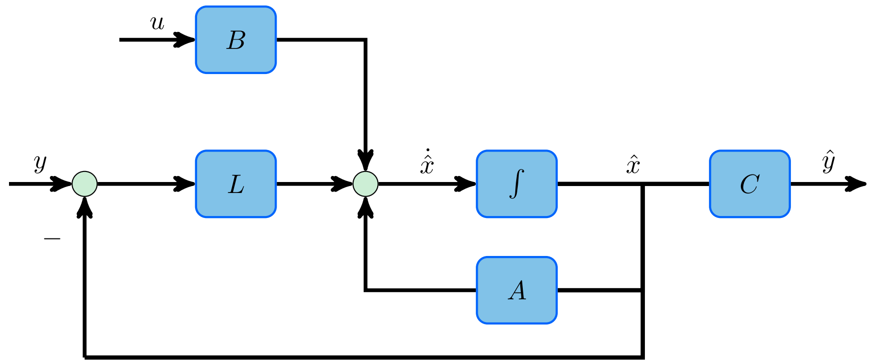


Figure 10: Observer Detailed Block Diagram

Let

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}),\ \hat{y} = C\hat{x}$$

where $L \in \mathbb{R}^{n \times p}$. Hence,

$$\dot{\hat{x}} = A\hat{x} + Bu + Ly - LC\hat{x} = (A - LC)\hat{x} + Bu + Ly$$

and

$$y - \hat{y} = C(x - \hat{x})$$

We call

$$e(t) = x(t) - \hat{x}(t)$$

the estimation error which satisfies

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} = Ax + Bu - (A - LC)\hat{x} - Bu - Ly \\ &= Ax - (A - LC)\hat{x} - LCx = (A - LC)x - (A - LC)\hat{x} \\ &= (A - LC)e \end{aligned}$$

It therefore follows that if we can choose the feedback matrix L to be such that the system matrix $(A - LC)$ has negative real parts, then

$$\hat{x} \to x, \text{ as } t \to \infty$$

(i.e. an *asymptotic estimate*) irrespective of the plant input u!

As we have already seen with pole placement, the gain matrix L of the full-state observer can be computed using any of the methods used to compute the control gain matrix K. We will assume that the system is completely observable. Therefore, the closed-loop eigenvalues of the observer can be placed at specified locations through the choice of L. For the control problem with full-state feedback, the closed-loop system matrix of interest is $A - BK$. Comparing that with the observer problem, the closed-loop system matrix is $A - LC$. The structure of those two matrices is similar; only the order of the unknown matrix differs between BK and LC.

Recall from **[510]** that the eigenvalues of a matrix and its transpose are the same. Hence, the observer problem can be formulated the same way as the control problem by considering the matrix $(A - LC)^\top = A^\top - C^\top L^\top$.

example. Consider the system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix} x(t) \end{aligned}$$

Suppose we want to place the poles of the observer at $\{-4, -4\}$. It is easy to check that the system is completely observable. Let $L = (\ell_1, \ell_2)$ be the unknown observer gain. Write the generic state estiamtion matrix

$$A - LC = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2}\ell_1 \\ 1 & -1 - \frac{1}{2}\ell_2 \end{bmatrix}$$

The characteristic polynomial of the observer is

$$\det(\lambda I - A + LC) = \lambda^2 + \left(2 + \frac{1}{2}\ell_2\right)\lambda + \frac{1}{2}\ell_2 + \frac{1}{2}\ell_1 + 1$$

Impose the polynomial equals the desired one

$$(\lambda + 4)^2 = \lambda^2 + 8\lambda + 16$$

Then we solve the linear system of equations in ℓ_1, ℓ_2 to get

$$\ell_1 = 18, \quad \ell_2 = 12$$

The resulting Luenberger observer is

$$\frac{d\hat{x}}{dt} = \begin{bmatrix} -1 & -9 \\ 1 & -7 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 18 \\ 12 \end{bmatrix} y(t)$$

6 M3: Additional Notes

In this section, I provide additional notes on topics including duality between controllability and observability, and discrete time controllability and observability.

6.1 Duality

The controllability and observability theorems we have stated so far are intimately related. Consider the system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

Recalling our knowledge about computing adjoints, we can write the dual representation as

$$\begin{aligned} -\dot{\tilde{x}}(t) &= A^*(t)\tilde{x}(t) + C^*(t)\tilde{u}(t) \\ \tilde{y}(t) &= B^*(t)\tilde{x}(t) + D^*(t)\tilde{u}(t) \end{aligned}$$

where here $\tilde{x}(t) \in \mathbb{C}^n, \tilde{u}(t) \in \mathbb{C}^p, \tilde{y}(t) \in \mathbb{C}^m$. The state transition matrix is

$$\Psi(t, \tau) = \Phi(\tau, t)^*$$

The minus sign on the dynamics essentially captures that the dual runs in reverse time. If we take the dual of the dual, we get

$$(A(\cdot), -B(\cdot), -C(\cdot), D(\cdot))$$

so that the original system is equal to the dual of the dual modulo a sign change for the state. And thus

$$(\mathcal{L}_r^*)^* = -\mathcal{L}_r, \; (\mathcal{L}_c^*)^* = -\mathcal{L}_c, \; (\mathcal{L}_o^*)^* = -\mathcal{L}_o$$

It turns out that controllability to zero on $[t_0, t_1]$ is the dual of unobservability on $[t_0, t_1]$ and vice versa.

Theorem 6.1 (Duality: controllability to zero versus unobservability.) The subspace of all states of $R(\cdot)$ that are controllable to zero (unobservable) on $[t_0, t_1]$ is the *orthogonal complement* of the subspace of all states of its dual $\tilde{R}(\cdot)$ that are unobservable (controllable to zero, resp.) on $[t_0, t_1]$. That is,

$$\mathrm{Im}(\mathcal{L}_c) = \mathrm{Ker}(\tilde{\mathcal{L}}_o)^\perp \quad \text{and} \quad \mathrm{Ker}(\mathcal{L}_o) = \mathrm{Im}(\tilde{\mathcal{L}}_c)^\perp$$

or equivalently

$$\mathrm{Im}(W_c[t_0, t_1]) = \mathrm{Ker}(\tilde{W}_o(t_0, t_1))^\perp \quad \text{and} \quad \mathrm{Ker}(W_o(t_0, t_1)) = \mathrm{Im}(\tilde{W}_c[t_0, t_1])^\perp$$

Proof. First, we have that x_0 of (A, B, C, D) is controllable to zero on $[t_0, t_1]$ if and only if there exists $u_{[t_0,t_1]}$ such that

$$x_0 = -\int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\; d\tau$$

or equivalently, if and only if $x_0 \in \mathrm{Im}(\mathcal{L}_c)$ where

$$\mathcal{L}_c : u_{[t_0,t_1]} \mapsto \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\; d\tau$$

Now, we know that

$$x_0 \text{ is unobservable on } [t_0, t_1] \iff x_0 \in \mathrm{Ker}(\mathcal{L}_o)$$

so that $\tilde{x}_0$ of $(-A^*, -C^*, B^*, D^*)$ is unobservable on $[t_0, t_1]$ if and only if $\tilde{\mathcal{L}}_0\tilde{x}_0 = 0$, or more precisely, if and only if

$$B(t)^*\Psi(t, t_0)\tilde{x}_0 = B^*(t)\Phi(t_0, t)^*\tilde{x}_0 = 0, \quad \forall t \in [t_0, t_1] \tag{17}$$

However, we know that

$$(A, B) \text{ controllable on } [t_0, t_1] \iff \mathrm{Im}(\mathcal{L}_c) = \mathbb{C}^n \iff \mathrm{Im}(\mathcal{L}_c\mathcal{L}_c^*) = \mathbb{C}^n \iff \det(W_c(t_0, t_1)) \neq 0$$

Hence, (17) is equivalent to $\tilde{x}_0 \in \mathrm{Ker}(W_c(t_0, t_1)) = \mathrm{Ker}(\mathcal{L}_c^*)$. We also know that $\mathrm{Im}(\mathcal{L}_c) = \mathrm{Ker}(\mathcal{L}_c^*)^\perp$, so that we have established $\mathrm{Im}(\mathcal{L}_c) = \mathrm{Ker}(\tilde{\mathcal{L}}_o)^\perp$ since $\mathcal{L}_c^* = \tilde{\mathcal{L}}_o$. □

Corollary 38. The system $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ is controllable (observable) on $[t_0, t_1]$ iff its dual is observable (controllable, resp.) on $[t_0, t_1]$.

6.2 Discrete Time Controllability and Reachability

Remark. The fundamental difference between CT and DT is that DT controllability to zero does not necessarily imply reachability from zero.

Definition 39. We say that the pair $(A(\cdot), B(\cdot))$ is controllable on $[k_0, k_1]$ iff for all (x_0, k_0) and for all (x_1, k_1) there exists a control sequence $u_{[k_0,k_1-1]} = (u(k_0), \ldots, u(k_1 - 1))$ that transfers the (x_0, t_0) to the (x_1, t_1).

Given the system $(A(\cdot), B(\cdot))$, we know that $u_{[k_0,k_1-1]}$ transfers x_0 to x_1 iff

$$x_1 = s(k_1, k_0, x_0, u_0) = \Phi(k_1, k_0)x_0 + \sum_{\ell=k_0}^{k_1-1} \Phi(k_1, \ell + 1)B(\ell)u(\ell)$$

This expression, in turn, shows that there will be such an input taking arbitrary x_0 to arbitrary x_1 if and only if the linear map

$$\mathcal{L}_r(k_0, k_1) : \mathcal{U}_d(k_0, k_1 - 1) \to \mathbb{C}^n : u_{[k_0,k_1-1]} \to \sum_{\ell=k_0}^{k_1-1} \Phi(k_1, \ell + 1)B(\ell)u(\ell)$$

is surjective. This map is the reachability map and since it is linear, we can invoke the matrix representation theorem to note that it has a matrix representation L_r given by

$$L_r(k_0, k_1) = \begin{bmatrix} B(k_1 - 1) & \Phi(k_1, k_1 - 1)B(k_1 - 2) & \cdots & \Phi(k_1, k_0 + 1)B(k_0) \end{bmatrix}$$

Unlike the CT case,

$$\Phi(k_1, k_0) = A(k_1 - 1)A(k_1 - 2) \cdots A(k_0)$$

has an inverse $\Phi(k_0, k_1) = (\Phi(k_1, k_0))^{-1}$ iff $\det(A(k)) \neq 0$ for all $k \in [k_0, k_1 - 1]$. Because of this fact, we have the following result.

Theorem 40. The following implications hold:

$$\begin{aligned}
&(A(\cdot), B(\cdot)) \text{ is controllable on } [k_0, k_1] \\
&\iff \forall x_1 \in \mathbb{C}^n,\ \exists u_{[k_0, k_1 - 1]} \text{ that steers } (0, k_0) \text{ to } (x_1, k_1) \text{ (reachable)} \\
&\implies \forall x_0 \in \mathbb{C}^n,\ \exists u_{[k_0, k_1 - 1]} \text{ that steers } (x_0, k_0) \text{ to } (0, k_1) \text{ (controllable to zero)}
\end{aligned}$$

where the last statement is actually an equivalence if $\det(A(k)) \neq 0$ for all $k \in [k_0, k_1 - 1]$.

Example 41. Show that the constant pair (A, b) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ b = e_2$$

is not controllable on $[0, 3]$ yet every state x_0 is driven to zero at $k = 3$. Indeed, this matrix A is nilpotent with $k = 3$ and hence with the zero control input every state goes to zero. On the other hand there are clearly states x_1 which are not reachable by any control.