AA/EE547: Wi 22

Module 2: Stability

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References. Chapter 8 and 9, **[JH]**; Chapter 4 and Chapter 7, **[C&D]**. review your notes on norms including (induced) matrix norms from [510]

Contents

Note: You can click on the links above under contents to bring you to the corresponding place in the pdf.

1 M2-RL1: Introduction to Stability

You may recall from an undergrad controls or signals class input-output stability. In our study of stability, we will start with state related stability concepts (stability in the sense of Lyapunov).

Intuition. Let's consider the following examples.

a. Continuous Time. Recall that the solution to

$$\dot{x} = -\lambda x$$

is

$$x(t) = x_0 e^{-\lambda t},$$

and if $\lambda > 0$ solution decays to zero, otherwise it blows up. This λ is an 'eigenvalue' for this scalar system, and its sign can be used to characterize a notion of stability.

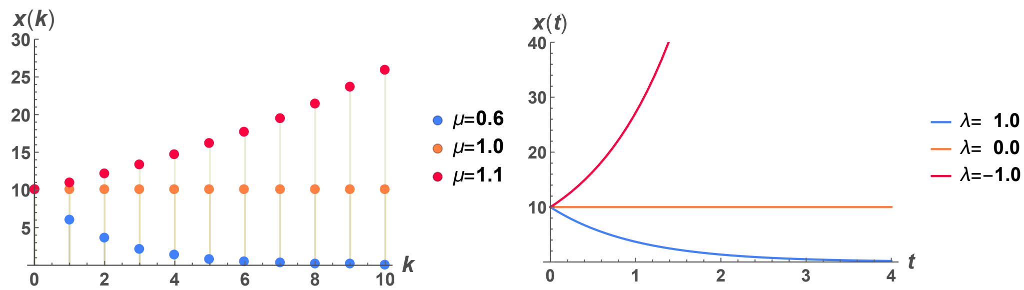
b. Discrete Time. Recall that the solution to

$$x_{k+1} = \mu x_k$$

is

$$x_k = \mu^k x_0$$

and if $|\mu| < 1$, then the solution decays to zero and otherwise, it blows up.



(a) discrete time (b) continuous time

Examples.

1. **Linear systems.** Consider the linear system $\dot{x} = Ax$.
   - If A is non-singular then $x^* = 0$ is the unique equilibrium.
   - If A is singular, then the null space defines a continuum of equilibria.
2. **Logistic Growth Model.** In population dynamics, the logistic growth model is common:

$$\dot{x} = f(x) = r\left(1 - \frac{x}{K}\right)x, \;\; r > 0, \;\; K > 0$$

The equilibria are determined by solving

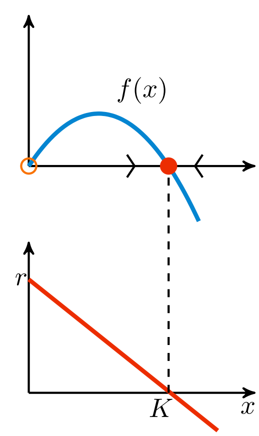
$$f(x) = r\left(1 - \frac{x}{K}\right)x = 0$$

Clearly

$$x^* = 0, \;\; x^* = K$$

are equilibrium. The state variable $x > 0$ denotes the population and K is the carrying capacity.When $x \in \mathbb{R}$ is scalar, stability can be determined from the sign of $f(x)$ around the equilibrium. In this example, $f(x) > 0$ for all $x \in (0, K)$ and $f(x) < 0$ for all $x > K$. Thus

   - $x = 0$ is an unstable equilibrium
   - $x = K$ is stable (in fact asymptotically so)



Notice that the second example is a **nonlinear system**. Local stability properties of an equilibrium x^* for a nonlinear system can be determined by linerizing the dynamics (i.e., the vector field $f(x)$) about the point x^*: i.e.,

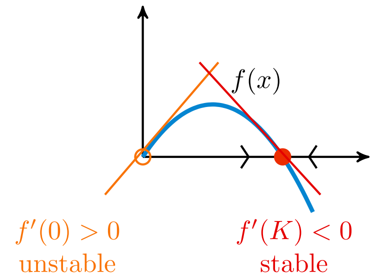
$$f(x^* + \delta\tilde{x}) = \underbrace{f(x^*)}_{=0} + \underbrace{Df\big|_{x=x^*}}_{=:A}\delta\tilde{x} + O(\delta^2)$$

so that the linearized model is given by

$$\dot{\tilde{x}} = A\tilde{x}$$

Stability of this system implies stability of the equilibrium x^* for the nonlinear dynamics.

Coming back to the nonlinear example of the logistic growth dynamics, we have that



1.1 Continuous Time

Recall that for a given linear system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned}$$

the zero input response is given by

$$x(t) = \Phi(t, t_0)x_0$$

where $\Phi(t, t_0)$ is the state transition matrix and $x(t_0) = x_0$. Note that

$$x_0 = 0 \Longrightarrow x(t) = 0\ \forall t$$

The points $x^* = 0$ is called the *equilibrium point*.

Definition 1 (Stable Equilibrium)**.** The following are characterizations of stability (in the sense of Lyapunov).

a. **Marginally Stable**: Consider the equilibrium point $x^* = 0$.

$$x^* \text{ is stable } \iff \forall\ x_0 \in \mathbb{R}^n,\ \forall t_0 \in \mathbb{R}^n,\ t \mapsto x(t) = \Phi(t, t_0)x_0 \text{ is bounded } \forall\ t \geq t_0.$$

Note: the effect of initial conditions does not grow unbounded with time (but it may grow temporarily during a transient phase).

b. **Asymptotic Stability**. Consider the equilibrium point $x^* = 0$.

$$x^* = 0 \text{ is asymptotically stable } \iff x_0 = 0 \text{ is stable and } x(t) = \Phi(t, t_0)x_0 \longrightarrow 0 \text{ as } t \to \infty.$$

Note: the effect of initial conditions eventually disappears with time.

c. **Exponential Stability**. Consider the equilibrium point $x^* = 0$.

$$x^* = 0 \text{ is exponentially stable } \iff \exists M, \alpha > 0 :\ \|x(t)\| \leq M\exp(-\alpha(t - t_0))\|x_0\|$$

We say an equilibrium point or the system is **unstable** if it is not marginally stable in the sense of Lyapunov. For such systems, the effect of initial conditions (may) grow over time (depending on the specific initial conditions).

Theorem 2 (Asymptotic Stability of Linear CT Systems)**.** The following claim holds:

$$x = 0 \quad \text{is asymptotically stable} \iff \Phi(t,0) \to 0 \quad \text{as } t \to \infty.$$

Proof. ($\Longleftarrow$) Observe that

$$x(t) = \Phi(t,t_0)x_0 = \Phi(t,0)\Phi(0,t_0)x_0$$

since $\Phi(t,0) \to 0$ as $t \to \infty$ then $\|\Phi(t,0)\| \to 0$ as $t \to \infty$ and

$$\|x(t)\| \leq \|\Phi(t,0)\|\|\Phi(0,t_0)\|\|x_0\|$$

Thus $x(t) \to 0$ as $t \to \infty$.
($\Longrightarrow$) We argue by contradiction: assume that $t \to \Phi(t,0)$ does not tend to zero as $t \to \infty$, i.e. $\exists\ i,j$ such that[1]

$$\Phi_{ij}(t,0) \not\to 0 \quad \text{as} \quad t \to \infty$$

Choose

$$x_0 = \left.\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} \quad \text{where 1in the } j\text{-th spot}$$

Thus, we have

$$x_i(t) = \Phi_{ij}(t,0) \not\to 0 \quad \text{as} \quad t \to \infty$$

contradicting the asymptotic stability of zero. □

A stability concept that is equivalent to exponential stability for LTV systems is **uniform asymptotic stability**.

Definition 3 (Uniform Asymptotic Stability)**.** We say that the zero solution of $\dot{x} = A(t)x(t)$ on $t \geq 0$ is uniformly asymptotically stable if and only if

a. $t \mapsto \Phi(t,t_0)$ is bounded on $t \geq t_0$ uniformly in $t_0 \in \mathbb{R}_+$, i.e.,

$$\exists\ k < \infty: \ \forall t_0 \in \mathbb{R}_+,\ \|\Phi(t,t_0)\| \leq k,\ \forall t \geq t_0$$

b. $t \mapsto \Phi(t,t_0)$ tends to zero as $t \to \infty$ uniformly in $t_0 \in \mathbb{R}_+$, i.e.,

$$\forall \varepsilon > 0,\ \exists T(\varepsilon) > 0: \ \forall t_0 \in \mathbb{R}_+, \|\Phi(t,t_0)\| \leq \varepsilon,\ \forall t \geq t_0 + T(\varepsilon) \tag{1}$$

In the LTI case, i.e. $\dot{x} = Ax$,

$$\text{asymptotic stability} \iff \text{exponential stability}$$

Indeed, $\Phi(t,t_0) = \exp(A(t-t_0))$ depends only on the elapsed time $t-t_0$, so that the zero solution of $\dot{x} = Ax$ is asymptotically stable if and only if the zero solution is uniformly asymptotically stable if and only if the zero solution is exponentially stable.

Notation. Recall that $\exists!$ means "there exists a unique".

[1] The notation Φ_{ij} denotes the (i,j) entry of the matrix Φ.

Proposition 4 (LTV Exponential Stability)**.**

$$A(\cdot) \text{ uniformly asymptotically stable } \iff A(\cdot) \text{ exponentially stable}$$

Proof. ($\Longleftarrow$) Suppose that $A(\cdot)$ is exponentially stable. Then, items a. and b. of Definition 3 hold with $k = m$ and $T(\varepsilon) > 0$ such that $\exp(-\alpha T(\varepsilon)) \leq \varepsilon m^{-1}$. Hence, we have that $A(\cdot)$ is uniformly asymptotically stable.

($\Longrightarrow$) Suppose that $A(\cdot)$ is uniformly asymptotically stable. Given any $T > 0$, we have that

$$\forall t \geq t_0,\ \exists!\ n \in \mathbb{N},\ \exists!\ s \in [0, T):\ t - t_0 = nT + s$$

Let $t_0 \in \mathbb{R}_+$ be arbitrary but fixed and for b., pick some $T(\varepsilon) > 0$ for $\varepsilon = 1/2$. Then, by (1), we have that

$$\|\Phi(s + t_0 + T, t_0)\| \leq 1/2 \quad \forall\ s \geq 0.$$

By the properties of the state transition matrix (cf Module 1), we know that

$$\Phi(s + t_0 + 2T, t_0) = \Phi(s + t_0 + 2T, t_0 + T)\Phi(t_0 + T, t_0)$$

This in turn implies that

$$\|\Phi(s + t_0 + 2T, t_0)\| \leq \|\Phi(s + t_0 + 2T, t_0 + T)\| \|\Phi(t_0 + T, t_0)\| \leq 2^{-2},$$

we have that (by induction),

$$\forall s \geq 0, \forall n = 1, 2, \ldots,\ \|\Phi(s + t_0 + nT, t_0)\| \leq 2^{-n} \qquad (**)$$

Pick $\alpha > 0$ s.t. $\exp(\alpha T) = 2$. Then,

$$\forall s \in [0, T), 1 \leq 2\exp(-\alpha s) \implies \forall s \in [0, T),\ 2^{-n} \leq 2\exp(-\alpha(s + nT))$$

Combining this with the upper bound on the norm of the state transition matrix in $(**)$, we have that

$$\forall s \in [0, T), \forall n = 1, 2, \ldots,\ \|\Phi(s + t_0 + nT, t_0)\| \leq 2\exp(-\alpha(s + nT))$$

Now, using a. and the fact that $1 \leq 2\exp(-\alpha s)$, we have that

$$\forall s \in [0, T),\ \|\Phi(s + t_0, t_0)\| \leq k \leq 2k\exp(-\alpha s)$$

so that

$$\forall t \geq t_0,\ \|\Phi(t, t_0)\| \leq 2k\exp(-\alpha(t - t_0))$$

□

1.2 Stability of Discrete Time Linear Systems

Consider the LTV discrete time system given by

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k + D_k u_k \end{aligned}$$

The zero state is again the equilibrium for this system—i.e., $x^* = 0$.

Recall from **Module 1** (this is a link, btw) that the solution solution and output are given by

$$\begin{aligned} x_k &= \Phi(k, k_0)x_0 + \sum_{\ell=k_0}^{k-1} \Phi(k, \ell + 1)B_\ell u_\ell \\ y_k &= C_k\Phi(k, k_0)x_0 + C_k\left(\sum_{\ell=k_0}^{k-1} \Phi(k, \ell + 1)B_\ell u_\ell\right) + D_k u_k \end{aligned}$$

Asymptotic stability can be characterized informally by the statement that every solution of $x_{k+1} = A_k x_k$ tends to zero as $k \to \infty$.

Definition 5 (Asymptotic Stability)**.** Consider $x_k \equiv 0$ (i.e., the zero solution of $x_{k+1} = A_k x_k$). The zero solution is asymptotically stable if and only if for all $x_0 \in \mathbb{R}^n$, for all $k_0 \in \mathbb{N}$,

a. $k \mapsto x_k = \Phi(k, k_0)x_0$ is bounded on $k \geq k_0$
b. $k \mapsto x_k = \Phi(k, k_0)x_0 \to 0$ as $k \to \infty$.

Note that any solution x_k on any $[k_0, k]$ is a finite set and hence, item b. $\implies$ item a.

Due to linearity, we get the following theorem.

Theorem 6 (Asymptotic Stability of Linear DT Systems)**.** Let $\det(A_k) \neq 0$ for all $k \in \mathbb{N}$. The zero solution of $x_{k+1} = A_k x_k$ on $k \geq 0$ is asymptotically stable if and only if $\Phi(k, 0) \to 0$ as $k \to \infty$.

Practice Problem. Try to prove this theorem by following the same proof structure as Theorem 2. Hint: not that since $\det(A_k) \neq 0$, $\forall k \geq k_0 \geq 0$, $\Phi(k, k_0) = \Phi(k, 0)\Phi(0, k_0)$.

Analogous to the continuous time case, exponential stability is a property of the system $x_{k+1} = A_k x_k$ which if possessed, guarantees that every solution of the system is bounded by a decaying exponential depending on the elapsed time $k - k_0$. Indeed, we have the following formal definition.

Definition 7 (DT Exponential Stability)**.** The zero solution of $x_{k+1} = A_k x_k$ on $k \geq 0$ is exponentially stable if and only if $\exists\ \rho \in [0, 1)$ and $m > 0$ such that for all $k_0 \in \mathbb{N}$,

$$\|\Phi(k, k_0)\| \leq m\rho^{k-k_0},\ \forall\ k \geq k_0$$

where the matrix norm is arbitrary.[2]

The following are important observations:

- The constants $\rho \in [0, 1)$ and $m > 0$ are fixed, meaning that they are independent of k_0. Further, the constant $\alpha \geq 0$ such that $\rho = \exp(-\alpha)$ is the exponential decay rate.
- An equivalent statement: the zero solution is exponentially stable if and only if

$$\exists \rho \in [0, 1),\ m > 0 :\ \forall (x_0, k_0) \in \mathbb{R}^n \times \mathbb{N},\ \|x_k\| \leq m\|x_0\|\rho^{k-k_0},\ \forall k \geq k_0$$

We say that the zero solution is uniformly asymptotically stable if and only if

a. $k \mapsto x_k = \Phi(k, k_0)x_0$ on $k \geq k_0$ is uniformly bounded (bound is independent of k_0)
b. $k \mapsto x_k = \Phi(k, k_0)x_0 \to 0$ uniformly as $k \to \infty$.

Theorem 8 (Equivalence of Asymptotic and Exponenital Stability)**.**

$$A(\cdot) \text{ is uniformly asymptotically stable } \iff A(\cdot) \text{ is exponentially stable}$$

Practice Problem. Prove this theorem. Hint: use the same method as in the CT case. In the necessity direction use a. and b. with $2\varepsilon = 1$ and pick $\rho \in [0, 1)$ such that $2\rho^K > 1$ so that $\|\Phi(k, k_0)\| \leq 2\ell\rho^{k-k_0}$ for all $k \geq k_0$.

[2] Recall from [510] that finite dimensional norms are equivalent.

2 M2-RL2: Spectral Conditions for Stability of LTI Systems

In the case of LTI systems, characterization of stability reduces to analyzing spectral conditions. For this we need to remind ourselves of some of the results from [510] (link to notes)on functions of a matrix.

Recall from [510] that we have the following result about representing any analytic function[3] as a sum of polynomials.

Let $A \in \mathbb{C}^{n\times n}$ and let $\text{spec}(A)$ denote the spectrum of A (containing distinct eigenvalues of A) with $p = |\text{spec}(A)|$. The minimal polynomial of A is given as above by

$$\psi_A(\lambda) = \prod_{i=1}^{p}(\lambda - \lambda_i)^{m_i}$$

Facts about functions of a matrix. Let $f(s)$ be any function of s analytic on the spectrum of A and $q(s)$ be a polynomial such that

$$f^{(k)}(\lambda_\ell) = q^{(k)}(\lambda_\ell)$$

for $0 \le k \le m_\ell - 1$ and $1 \le \ell \le p$. Then

$$f(A) = q(A)$$

where p is the number of distinct roots of the characteristic polynomial $\chi_A(s)$ and

$$m_k = \min\{\mu \ge 1 : \mathcal{N}((A - \lambda_k I)^\mu) = \mathcal{N}((A - \lambda_k I)^{\mu+1})\}$$

i.e., m_k is the ascent of $A - I\lambda_k$.

In fact, if $m = \sum_{i=1}^{p} m_i$ then

$$q(s) = a_1 s^{m-1} + a_2 s^{m-2} + \cdots + a_m s^p$$

where $a_1, a_2, \ldots, a_n$ are functions of

$$(f(\lambda_1), f^{(1)}(\lambda_1), f^{(2)}(\lambda_1), \ldots, f^{(m_1)}(\lambda_1), f(\lambda_2), \ldots)$$

and hence

$$f(A) = a_1 A^{m-1} + \cdots + a_m A^0 = \sum_{\ell=1}^{p}\sum_{k=0}^{m_\ell - 1} q_{k,\ell}(A) f^{(k)}(\lambda_\ell)$$

where $q_{k,\ell}$'s are polynomials independent of f.

This leads to the following theorem.

Theorem 9 (General Form of $f(A)$)**.** Let $A \in \mathbb{C}^{n\times n}$ have a minimal polynomial ψ_A given by

$$\psi_A(s) = \prod_{k=1}^{p}(s - \lambda_k)^{m_k}$$

Let the domain Δ contain $\text{spec}(A)$, then for any analytic function $f : \Delta \to \mathbb{C}$. we have

$$f(A) = \sum_{k=1}^{p}\sum_{\ell=0}^{m_k - 1} f^{(\ell)}(\lambda_k) q_{k,\ell}(A)$$

where $q_{k,\ell}$'s are polynomials independent of f.

[3] An *analytic function* is a function that is locally given by a convergent power series—i.e., f is real analytic on an open set U in the real line if for any $x_0 \in U$ one can write

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $a_n \in \mathbb{R}$ for each n and the series is convergent to $f(x)$ for $x$in a neighborhood of x_0.

Recalling our derivation of functions of matrices from [510], we can show that

$$\exp(tA) = \sum_{k=1}^{p} \sum_{\ell=0}^{m_k-1} t^\ell \exp(\lambda_k t) p_{k\ell}(A)$$

This gives rise to the following stability condition:

Proposition 10 (Continuous Time)**.** Consider the differential equation $\dot{x} = Ax$, $x(0) = x_0$. From the above expression:

$$\{\exp(At) \to 0 \quad \text{as} \quad t \to \infty\} \iff \{\forall \lambda_k \in \mathrm{spec}(A),\ \mathrm{Re}(\lambda_k) < 0\}$$

and

$$\{t \mapsto \exp(At) \text{ is bounded on } \mathbb{R}_+\} \iff \left\{ \begin{array}{ll} \forall \lambda_k \in \mathrm{spec}(A), & \mathrm{Re}(\lambda_k) < 0 \ \& \\ m_k = 1 \text{ when} & \mathrm{Re}(\lambda_k) = 0 \end{array} \right\}$$

We have a similar situation for discrete time systems:

$$\forall\ \nu \in \mathbb{N},\ A^\nu = \sum_{k=1}^{p} \sum_{\ell=1}^{m_k-1} \nu(\nu-1)\cdots(\nu-\ell+1)\lambda_k^{\nu-\ell} p_{k\ell}(A)$$

The above gives rise to the following stability condition:

Proposition 11 (Discrete Time)**.** Consider the discrete time system $x(k+1) = Ax(k)$, $k \in \mathbb{N}$, with $x(0) = x_0$. Then for $k \in \mathbb{N}$, $x(k) = A^k x_0$. From the above equation, we have that

$$\left\{A^k \to 0 \text{ as } k \to \infty\right\} \iff \{\forall\ \lambda_i \in \mathrm{spec}(A),\ |\lambda_i| < 1\}$$

and

$$\left\{k \to A^k \text{ is bounded on } \ \mathbb{N}_+\right\} \iff \left\{ \begin{array}{ll} \forall \lambda_i \in \mathrm{spec}(A), & |\lambda_i| \leq 1 \ \& \\ m_i = 1 \ \text{ when} & |\lambda_i| = 1 \end{array} \right\}$$

2.1 LTI: Asymptotic Stability is Equivalent to Exponential Stability

Coming back to the CT case, we will show that asymptotic stability is equivalent to exponentially stable using the function of matrix expansion.

Let's prove the claim in Proposition 10:

Claim 1.

$$\dot{x} = Ax \text{ is exponentially stable } \iff \mathrm{spec}(A) \subset \mathbb{C}_-^\circ$$

Proof. The state transition matrix for an LTI system is

$$\Phi(t, t_0) = \exp(A(t - t_0))$$

And from above we know that

$$\exp(At) = \sum_{k=1}^{p} \pi_k(t) \exp(\lambda_k t),\ \{\lambda_k\}_1^p = \mathrm{spec}(A)$$

where π_k are some matrix polynomials in t. Hence, by taking matrix norms, we have that

$$\|\exp(At)\| \leq \sum_{k=1}^{p} \|\pi_k(t)\| \exp(\mathrm{Re}(\lambda_k)t) \leq \sum_{k=1}^{p} p_k(t) \exp(\mathrm{Re}(\lambda_k)t) \leq p(t)\exp(-\mu t)$$

where $p_k(t)$ are polynomials such that $\|\pi_k(t)\| \leq p_k(t)$, $p(t) = \sum_{k=1}^{p} p_k(t) \geq 0$ and

$$\mu = -\max\{\mathrm{Re}(\lambda) : \ \lambda \in \mathrm{spec}(A)\}$$

Since a polynomial is growing slower than any growing exponential we have

$$\forall \varepsilon > 0, \ \exists \ m(\varepsilon) > 0 : \ 0 \leq |p(t)| \leq m \exp(\varepsilon t), \ \forall t \geq 0$$

Hence combing this with the above bound on $\|\exp(At)\|$, we have that

$$\forall \varepsilon > 0 \ \exists \ m(\varepsilon) > 0 : \ \|\exp(At)\| \leq m \exp(-(\mu - \varepsilon)t) \ \forall t \geq 0$$

Then, if $\mathrm{spec}(A) \subset \mathbb{C}_-^\circ$, by the above $\mu > 0$. Hence picking $\varepsilon \in (0, \mu)$ we have that

$$\|\Phi(t, t_0)\| \leq m \exp(-\alpha(t - t_0))$$

with $\alpha = \mu - \varepsilon > 0$. On the other hand if $\mathrm{spec}(A)$ is not included in $\mathbb{C}_-^\circ$, then by the polynomial expansion for $\exp(At)$, $\exp(At)$ does not tend to the zero matrix as $t \to \infty$ and the zero solution is not exponentially stable. □

In the subsequent subsections we will explore this notion of assessing stability of linear system via the spectral properties of the dynamics A by way of two important applications of the theory:

1. **M2-RL2a** Numerical Integration: Choosing a stepsize to ensure stability.
2. **M2-RL2b** Nonlinear system stability by linearizing.

2.2 M2-RL2a: Application to Numerical Integration

Suppose we are given $\dot{x} = Ax$, $x(0) = x_0$, $A \in \mathbb{C}^{n \times n}$, $x \in \mathbb{C}^n$. Call $t \mapsto x(t)$ the exact solution $x(t) = \exp(At)x_0$. Note that $t \mapsto x(t)$ is analytic in t. Call $(\xi_0, \xi_1, \ldots)$ the sequence of computed values.

There are many numerical integration schemes. We will focus on two first order schemes: forward and backward Euler. These two schemes connect to stability very nicely.

Scheme 1: Forward Euler Method. For small $h > 0$, we have for any $t_k \in \mathbb{R}_+$,

$$\begin{aligned} x(t_k + h) &= x(t_k) + h\dot{x}(t_k) + \mathrm{O}(h^2) \\ &= x(t_k) + hAx(t_k) + \mathrm{O}(h^2) \end{aligned}$$

In other words, we have approximately

$$x(t_k + h) \simeq (I + hA)x(t_k)$$

So if we perform repeatedly this step starting at $t_0 = 0$, we have the computed sequence $\{\xi_i\}_0^\infty$ by

$$\xi_m = (I + hA)^m x_0, \ m = 0, 1, 2, \ldots$$

From the spectral mapping theorem and the above equation for ξ_m we have the following.

Example. Consider the scalar dynamical system

$$\dot{x}(t) = \lambda x(t)$$

with $\lambda \in \mathbb{C}$. Then the equation is stable if $\mathrm{Re}(\lambda) \leq 0$. In this case the system is exponentially decaying

$$\lim_{t\to\infty} x(t) = 0$$

When is the numerical solution x_i also decaying—i.e., when is $\lim_{i\to\infty} x_i = 0$? First, the **forward Euler** numerical solution is given by the following recursion where h is the step-size:

$$x_{k+1} = x_k + h\lambda x_k$$

Iterating the recursion, we have that

$$x_{k+1} = (1 + h\lambda)^{k+1} x_0$$

The solution is decaying (stable) if $|1 + h\lambda| \leq 1$. That is, for any step-size choice $h > 0$ such that

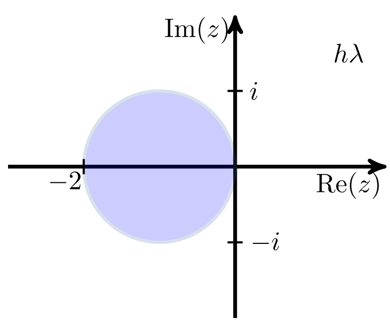
$$(1 + h\mathrm{Re}(\lambda))^2 + \mathrm{Im}(\lambda)^2 \leq 1$$

$$\begin{aligned} 1 + 2h\mathrm{Re}(\lambda)) + h^2\mathrm{Re}(\lambda))^2 + h^2\mathrm{Im}(\lambda)^2 \leq 1 &\iff 2h\mathrm{Re}(\lambda)) + h^2(\mathrm{Re}(\lambda))^2 + \mathrm{Im}(\lambda)^2) \leq 0 \\ &\iff h(2\mathrm{Re}(\lambda) + h|\lambda|^2) \leq 0 \\ &\iff h \leq -2\frac{\mathrm{Re}(\lambda)}{|\lambda|^2} \\ &\iff 0 \leq h \leq \frac{2|\mathrm{Re}(\lambda)|}{|\lambda|^2}, \quad \text{when system is stable—i.e., } \lambda < 0. \end{aligned}$$

Another way to think about this is letting $z = \lambda h$ so that we have the following constraint:

$$1 + 2x + x^2 \leq 1$$

which gives us the circle shown in the figure below (right).



Forward Euler

Fact 12. Suppose $\mathrm{spec}(A) \subset \mathbb{C}_-^\circ$ (equivalently, the origin is exponentially stable). Let h_0 be the largest positive h such that

$$\max_{\lambda\in\mathrm{spec}(A)} |1 + h\lambda| = 1$$

Under these conditions,

1. The iterates ξ_m are such that $\{\xi_m\}_0^\infty \to 0$ exponentially for all ξ_0 if and only if $h \in (0, h_0)$.
2. If $h > h_0$, then for almost all x_0, the sequence of computed values $\{\xi_k\}_0^\infty$ is such that $\{\|\xi_m\|\}_0^\infty$ grows exponentially.

Interpretation. Even if $\mathrm{spec}(A) \subset \mathbb{C}_-^\circ$ (and hence the exact solution $x(t) \to 0$ exponentially), for $h > h_0$, for almost all x_0, the sequence of computed vectors $\{\|\xi_m\|\}_0^\infty$ blows up. It is for this reason that in practice we often prefer the backward Euler method.

Scheme 2: Backward Euler. For small $h > 0$, we have that for any $t_k \in \mathbb{R}$,

$$\begin{aligned} x(t_k) &= x(t_k + h) - h\dot{x}(t_k + h) + \mathrm{O}(h^2) \\ &= x(t_k + h) - hAx(t_k + h) + \mathrm{O}(h^2) \end{aligned}$$

Thus we have approximately

$$x(t_k + h) \simeq (I - hA)^{-1} x(t_k)$$

So if we perform repeatedly this step, starting from $t_0 = 0$, we get the computed sequence $\{\xi_i\}_0^\infty$ given by

$$\xi_m = (I - hA)^{-m} x_0, \;\; m = 0, 1, 2, \ldots$$

Now, the spectrum of $(I - hA)^{-1}$ is $\{(1 - h\lambda_i)^{-1}\}_{i=1}^{\sigma}$. Hence by the above expression for ξ_m, we have

$$\begin{aligned} &\xi_m \to 0 \text{ as } m \to \infty \\ &\iff \forall \lambda_i \in \operatorname{spec}(A), \;\; |(1 - h\lambda_i)^{-1}| < 1 \\ &\iff \forall \lambda_i \in \operatorname{spec}(A), \;\; |1 - h\lambda_i| > 1 \end{aligned}$$

Note that if $\mathrm{Re}(\lambda_i) < 0$, then $|1 - h\lambda_i| > 1$, since $h > 0$. Thus we have shown the following result.

Example. Consider again the scalar linear dynamical system given by

$$\dot{x}(t) = \lambda x(t)$$

with $\lambda \in \mathbb{C}$. Then the equation is stable if $\mathrm{Re}(\lambda) \leq 0$. In this case the system is exponentially decaying

$$\lim_{t \to \infty} x(t) = 0$$

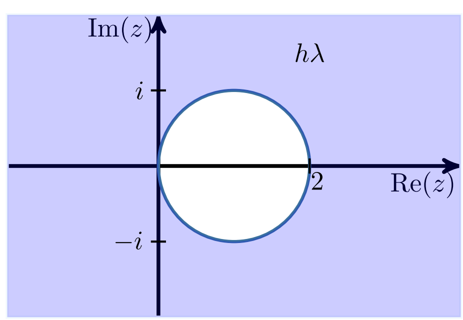
When is the numerically solution x_i also decaying, $\lim_{i \to \infty} x_i = 0$?

$$x_{i+1} = x_i + h\lambda x_{i+1} \iff x_{i+1} = \left(\frac{1}{1 - h\lambda}\right)^{i+1} x_0$$

The solution is decaying (stable) if $|1 + h\lambda| \geq 1$. Indeed, letting $z = h\lambda$ the above constraint is basically saying

$$1 + 2z + z^2 \geq 1$$

which is illustrated in the figure below.



Backward Euler

Fact 13. If $\operatorname{spec}(A) \subset \mathbb{C}_-^\circ$, then for all $h > 0$, for all $x_0 \in \mathbb{C}^n$, the computed sequence $\{\xi_m\}_0^\infty$ obtained via backward Euler goes to zero exponentially.

This is very important in practice, because if h is unfortunately chosen too large the computed sequence may lose accuracy but at least it will never blow up!

2.3 M2-RL2b: Stability of Nonlinear Systems

Last time we talked about internal or Lyapunov stability. Our focus was on linear dynamical systems but this is an important concept for nonlinear systems as well.

Consider the nonlinear system

$$\dot{x} = f(x) \tag{2}$$

where $f \in C^1(\mathbb{R}^n, \mathbb{R})$, $V \in C^1(\mathbb{R}^n, \mathbb{R})$. The flow of (2)—i.e., ϕ_t—is the solution of the ODE starting from the initial state x at time t_0. An equilibrium (or critical) point is a point x^* such that

$$\dot{x} = f(x^*) = 0.$$

As we saw last time, the linearization around an equilibrium point x^* is given by

$$\dot{x} = Df(x^*)x. \tag{3}$$

Notation: Let $B_\delta(x)$ denote the δ-radius ball around x defined using the Euclidean distance—i.e.,

$$B_\delta(x) := \{y \in \mathbb{R}^n | \ \|x - y\|_2 \leq \delta\}.$$

Recall from Module 1 that $\phi_t(x)$ denotes the flow of the ODE given in (2) and its definition is summarized below.

Definition 14 (Flow)**.** The state of the system (2) at time t starting from x at time zero (without loss of generality) is called the flow and is denoted $\phi_t(x)$.

Definition 15 (Stability)**.** An equilibrium point x^* of (2) is uniformly stable if for all $\varepsilon > 0$ and $t_0 \geq 0$, there exists a $\delta(\varepsilon) > 0$ such that for all $x \in B_\delta(x^*)$ and $t \geq t_0$ we have that

$$\phi_t(x) \in B_\varepsilon(x^*).$$

Moreover, an equilibrium point x^* is asymptotically stable if it is stable and if there exists a $\delta > 0$ such that

$$\forall x \in B_\delta(x^*), \quad \lim_{t\to\infty} \phi_t(x) = x^*.$$

The above definition in words is stating that a a point is stable if for all initial conditions in a neighborhood around the equilibrium x^*, the state of the system (flow) remain nearby. Further, stability is upgraded (strengthened) to asymptotic stability if the flow asymptotically converges to the equilibrium. Figure 1 provides an illustration.

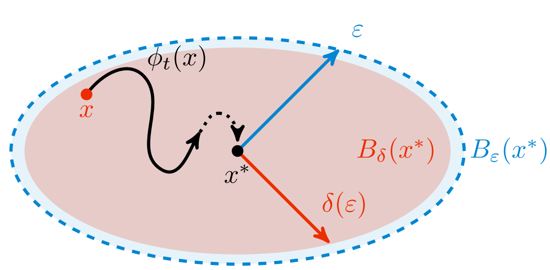


Figure 1: Illustration of Definition 15: as illustrated by the black line, the flow $\phi_t(x)$ from x eventually ends up in $B_\varepsilon(x^*)$, and the dashed lined illustrates asymptotic stability (i.e., $\lim_{t\to\infty} \phi_t(x) = x^*$).

Lyapunov Functions for Nonlinear Stability. The existence of a Lyapunov function is a one way to prove stability. For a dynamical system, $\dot{x} = f(x)$ with $x^* = 0$ as an equilibrium point, a scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function if it is C^1, locally positive definite, and $\dot{V} \leq 0$ where $\dot{V}$ is the derivative of the scalar function $V(x)$ along trajectories (the solution) $\phi_t(x)$ and is given by

$$\dot{V}(x) = \frac{d}{dt}V(\phi_t(x)) = \frac{\partial V(\phi_t)}{\partial \phi_t}\frac{d}{dt}\phi_t(x) = DV(x)f(x).$$

Positive definite means

$$V(z) \geq 0,$$

all sublevel sets are bounded (i.e. $V(z) \to \infty$ as $z \to \infty$), and

$$V(z) = 0 \iff z = 0.$$

Recall that sublevel sets of a function V are defined as follows: the α-sublevel set of V is given by

$$\{x \in \mathbb{R}^n |\ V(x) \leq \alpha\}.$$

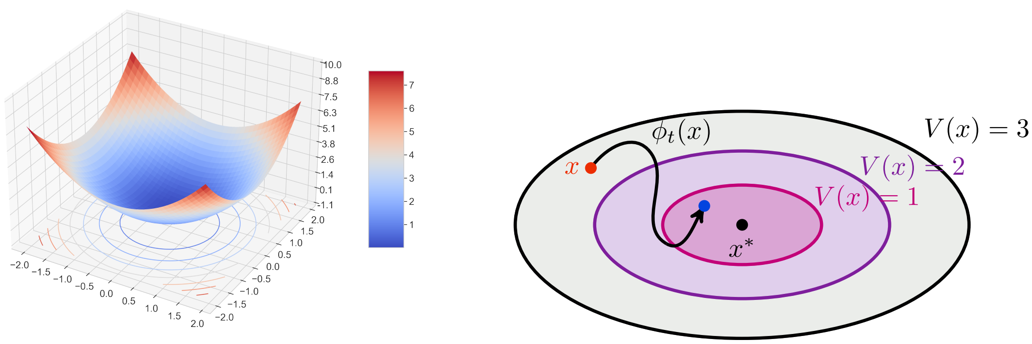


Figure 2: Example: (left) Illustration of a "toy" Lyapunov function and its contours; (right) Visualization of level sets $V(x) = 1$, $V(x) = 2$, and $V(x) = 3$ for a Lyapunov function V. If a state trajectory enters one of these sets, it has to stay inside it since $\dot{V}(x) \leq 0$.

Theorem 16 (Lyapynov Theorem)**.** Consider the dynamical system defined by $f \in C^1(\mathbb{R}^n, \mathbb{R})$. Let W be an open subset of $\mathbb{R}^n$ containing the equilibrium point x^*—i.e., $f(x^*) = 0$. Suppose that there exists a real-valued function $V \in C^1$ such that $V(x^*) = 0$ and $V(x) > 0$ when $x \neq x^*$. Then, the following implications hold:

a. $\dot{V}(x) \leq 0,\ \forall x \in W \implies x^*$ is stable.
b. $\dot{V}(x) < 0,\ \forall x \in W \backslash \{x^*\} \implies x^*$ is asymptotically stable.
c. $\dot{V}(x) > 0,\ \forall x \in W \backslash \{x^*\} \implies x^*$ is unstable.

It can be difficult to construct a Lyapunov function for nonlinear systems. Hence, an alternative is to assess stability in terms of the local linearized dynamics about equilirbium points (cf. Hartman Grobman Theorem).

2.4 Stability of Linearized Systems

Consider a general non-linear system

$$\dot{x} = f(x),\ x \in \mathbb{R}^n$$

with an equilibrium point x^* such that $f(x^*) = 0$. Recall that the local linearization around x^* is given by

$$\dot{\tilde{x}} = A\tilde{x}$$

with $\tilde{x} = x - x^*$ and $A := Df(x^*)$. The following theorem is the celebrated Hartman-Grobman theorem which states that trajectories of the nonlinear system are "equivalent" to trajectories of the linearization in a neighborhood of an equilibrium, and hence we can assess (local) stability of the nonlinear system by assessing stability of the linearized system.[4]

Theorem 17 (Hartman-Grobman)**.** Consider a nonlinear dynamical system $\dot{x} = f(x)$ with an equilibrium point x^* (i.e. $f(x^*) = 0$). If the linearization of the system $A := D_x f(x)|_{x=x^*}$ has no zero or purely imaginary eigenvalues then there exists a homeomorphism (i.e., a continuous map with a continuous inverse) from a neighborhood U of x^* into $\mathbb{R}^n$,

$$h : U \to \mathbb{R}^n,$$

taking trajectories of the nonlinear system $\dot{x} = f(x)$ and mapping them onto those of $\dot{\tilde{x}} = A\tilde{x}$. In particular, we have that x^* maps to the equilibrium of the linearized system—i.e., $h(x^*) = 0$.

The above theorem directly translates to the following corollary.

Corollary 18. Suppose that $f \in C^2(\mathbb{R}^n, \mathbb{R})$. If the linearized system is exponentially stable, then there exists a ball $B \subset \mathbb{R}^n$ around x^* and constants $c, \lambda > 0$ such that for every solution $x(t)$ to the nonlinear system that starts at $x(t_0) \in B$, we have

$$\|x(t) - x^*\| \le ce^{-\lambda(t-t_0)}\|x(t_0) - x^*\|.$$

This means that the properties of the linearized system are preserved in the nonlinear system.

Conversely, instability is also a property that is preserved.

Theorem 19. Suppose that $f \in C^2(\mathbb{R}^n, \mathbb{R})$. If the linearized system is unstable, then there are solutions that start arbitrarily close to x^* but do not converge to this point as $t \to \infty$.

Caution: marginal stability is not a transferable property.

Example 20 (Marginal Stability is Not Transferable)**.** The two systems

$$\dot{x} = \pm x^3$$

(one for "+" one for "−") have the same local linearization $\dot{\tilde{x}} = 0$ around $x^* = 0$ which is only marginally stable. However $\dot{x} = -x^3$ is such that x always converges to zero, while for $\dot{x} = x^3$, x always diverges away from the equilibrium point.

2.4.1 Application to Inverted Pendulum

Let's consider the inverted pendulum. I encourage you to play around with this in Python as well; a Jupyter Notebook is provided here on the course website.

The inverted pendulum has dynamics

$$m\ell^2\ddot{\theta} = mg\ell\sin\theta - b\dot{\theta} + u$$

Letting $x_1 = \theta$ and $x_2 = \dot{\theta}$, we have that

$$\begin{aligned}\dot{x}_1 &= \dot{\theta} = x_2\\ \dot{x}_2 &= \ddot{\theta} = \frac{g}{\ell}\sin(x_1) - \frac{b}{m\ell^2}x_2 + \frac{1}{m\ell^2}\tau\end{aligned}$$

[4] If you are interested in learning more about nonlinear systems, I suggest Shankar Sastry's book "Nonlinear Systems" [1].

And suppose that $b/(2m\ell) < g$ and that we observe $x_1 = \theta$, the angle. That is,

$$y(t) = g(x,u) = x_1$$

Consider the equilibrium resulting from applying $u^* = \tau$. There are two equilibrium of this system

$$\{x|\ x_1 = \pi \text{ or } x_1 = 0\}$$

Now, we linearize by computing

$$A := D_x f(x,u)|_{x^*,u^*} = \begin{bmatrix} 0 & I \\ \frac{g}{\ell}\cos(x_1) & -\frac{b}{m\ell^2} \end{bmatrix}\Big|_{x=x^*,u=u^*},\ B := D_u f(x,u)|_{x^*,u^*},\ C := D_x g(x,u)|_{x^*,u^*}$$

The local linearization of this system around the equilibrium point for which $x_1 = \pi$ is given by

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + Bu \\ \tilde{y} &= C\tilde{x} \end{aligned} \tag{4}$$

where $\tilde{x} := \delta x$ and with

$$A := \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix},\ B := D_u f(x,u)|_{x_1=\pi,u=\tau} = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix},\ C := D_x g(x,u)|_{x_1=\pi,u=\tau} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The eigenvalues of A are given by

$$\det(\lambda I - A) = \lambda\left(\lambda + \frac{b}{m\ell^2}\right) + \frac{g}{\ell} \iff \lambda = -\frac{b}{2m\ell^2} \pm \sqrt{\frac{b}{2m\ell^2} - \frac{g}{\ell}}$$

So, the linearized system is exponentially stable. Indeed, since $b/(2m\ell) < g$, the λ's have negative real part.

What about for the other equilibrium $x^* = 0$? For this case, the A matrix is given by

$$A := \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix}$$

so that the eigenvalues in this case are

$$\lambda = -\frac{b}{2m\ell^2} \pm \sqrt{\frac{b}{2m\ell^2} + \frac{g}{\ell}}.$$

Hence, we deduce

$$-\frac{b}{2m\ell^2} + \sqrt{\frac{b}{2m\ell^2} + \frac{g}{\ell}} > 0 \implies \text{unstable}$$

3 M2-RL3: Lyapunov Stability via Lyapunov's Equation

References. As a reminder, you can refer to the following textbook references:
- Chapter 8 **[JH]**;
- Chapter 7/7d, **[C&D]**;
- [510]: review your notes on norms including (induced) matrix norms.

Recall from **M2-RL2** that we defined the concept of a Lypunov function and used it as a certificate for stability (cf. Theorem 16). We noted that for nonlinear systems it can be difficult to construct a Lyapunov function. There is a lot of guess work and intuition that can go into it and some computational tools such as sums-of-squares polynomial tools to generate Lyapunov functions.[5]

For linear systems, it turns out that Lyapunov functions take the form

$$V(z) = z^\top P z$$

for some positive definite symmetric matrix $P \succ 0$.

Recall the definition of a positive definite matrix (cf. [510] lecture notes for more detail).

Definition 21 (Positive Definite Matrix)**.** A matrix $Q \in \mathbb{R}^{n\times n}$ is *positive semi-definite* (PSD) if

$$x^\top Q x \geq 0, \quad \forall\ x \in \mathbb{R}^n/\{0\}$$

and it is called positive definite (PD) if the inequality is strict.

Note that in general we apply this definition to any matrix $Q \in \mathbb{R}^{n\times n}$ as stated in the definition, meaning that Q does not need to be symmetric for this definition to apply. However, in this class we define Lyapunov functions in terms of positive definite and symmetric matrices—i.e., where $Q = Q^\top \succ 0$.

The following proposition (cf. [510] notes) is also useful to recall.

Proposition 22. Consider a symmetric matrix $Q = Q^\top \in \mathbb{R}^{n\times n}$. The following are equivalent:

a. The matrix Q is positive definite—i.e., $Q \succ 0$.
b. The spectrum of Q contains only strictly positive real numbers—i.e., $\mathrm{spec}(Q) \subset \mathbb{R}_{>0}$.
c. The determinants of all upper left submatrices of Q are positive.
d. There exists an $n \times n$ non-singular real matrix H such that $Q = H^\top H$.

For a positive definite symmetric matrix $Q = Q^\top \succ 0$, we also have that

$$0 < \lambda_{\min}(Q)\|x\|^2 \leq x^\top Q x \leq \lambda_{\max}(Q)\|x\|^2, \quad \forall x \neq 0.$$

The Lyapunov stability theorem provides an alternative condition to check whether or not the continuous-time homogeneous LTI system

$$\dot{x} = Ax \tag{5}$$

is asymptotically stable.

The equation

$$A^\top P + PA + Q = 0$$

is called the **Lyapunov equation**. Here, the matrices $P, Q \in \mathbb{R}^{n\times n}$ are symmetric. For a linear system $\dot{x} = Ax$, if

$$V(z) = z^\top P z$$

then

$$\dot{V}(z) = (Az)^\top P z + z^\top P (Az) = -z^\top Q z$$

That is, if $z^\top P z$ is the (generalized) energy, then $z^\top Q z$ is the associated (generalized) dissipation.

If $P \succ 0$, then the sublevel sets[6] of this function are ellipsoids and bounded. Further, we have that

$$V(z) = z^\top P z = 0 \iff z = 0.$$

If $P \succ 0, Q \succeq 0$, then all the trajectories of $\dot{x} = Ax$ are bounded (i.e., $\mathrm{Re}(\lambda_i) \leq 0$ and if $\mathrm{Re}(\lambda_i) = 0$, then λ_i corresponds to a Jordan block of size one). Further, the ellipsoids $\{z|\ z^\top P z \leq a\}$ are invariant sets.

[5]If you are interested in exploring this further there is a nice discussion and set of colab notebooks available here.
[6]i.e., $\{x|\ V(x) < a\}$

Theorem 23. The following conditions are equivalent:

a. The system (5) is asymptotically (equivalently exponentially) stable.
b. All the eigenvalues of A have strictly negative real parts.
c. For every symmetric positive definite matrix $Q = Q^\top \succ 0$, there exists a unique solution P to the *Lyapunov equation*
$$A^\top P + PA = -Q.$$
Moreover, P is symmetric and positive-definite—i.e., $P = P^\top \succ 0$—and is given by
$$P = \int_0^\infty e^{A^\top t} Q e^{At}\, dt.$$
d. There exists a symmetric positive-definite matrix $P = P^\top \succ 0$ for which the following Lyapunov matrix inequality holds:
$$A^\top P + PA < 0$$

Proof. Proof Sketch. We have already seen that items a and b are equivalent (cf. **M2-RL2**). To show that b implies c, we need to show that
$$P = \int_0^\infty e^{A^\top t} Q e^{At}\, dt$$
is the unique solution to $A^\top P + PA = -Q$. This can be done by showing that the following are true:

1. the integral is well-defined (i.e. finite),
2. P as defined solves $A^\top P + PA = -Q$,
3. P as defined is symmetric and positive definite, and
4. no other matrix solves the equation.

Indeed, let us show each of these properties holds.

1. This follows from exponential stability—i.e.
$$\|e^{A^\top t} Q e^{At}\|_{\text{op}} \to 0$$
exponentially fast as $t \to \infty$. Hence, the integral is absolutely convergent.
2. We simply need to check by direct verification. Plugging P into the left-hand side of the Lyapunov equation, we have that
$$A^\top P + PA = \int_0^\infty A^\top e^{A^\top t} Q e^{At} + e^{A^\top t} Q e^{At} A\, dt$$
We also have that
$$\frac{d}{dt}\left(e^{A^\top t} Q e^{At}\right) = A^\top e^{A^\top t} Q e^{At} + e^{A^\top t} Q e^{At} A$$
so that
$$A^\top P + PA = \int_0^\infty \frac{d}{dt}\left(e^{A^\top t} Q e^{At}\right)\, dt = \left(e^{A^\top t} Q e^{At}\right)\Big|_{t=0}^{\infty} = \left(\lim_{t\to\infty} e^{A^\top t} Q e^{At}\right) - e^{A^\top 0} Q e^{A0}$$
And, the right-hand side is equal to $-Q$ since $\lim_{t\to\infty} e^{At} = 0$ (by asymptotic stability) and $e^{A0} = I$.
3. This also follows by direct computation. Symmetry easily follows:
$$P^\top = \int_0^\infty (e^{A^\top t} Q e^{At})^\top\, dt = \int_0^\infty \left(e^{At}\right)^\top Q^\top \left(e^{A^\top t}\right)^\top\, dt = \int_0^\infty e^{A^\top t} Q e^{At}\, dt = P$$
To check positive definiteness, pick an arbitrary vector z and compute:
$$z^\top P z = \int_0^\infty z^\top e^{A^\top t} Q e^{At} z\, dt = \int_0^\infty w(t)^\top Q w(t)\, dt$$

where $w(t) = e^{At}z$. Since Q is positive definite, we get that $z^\top Pz \geq 0$. Moreover

$$z^\top Pz = 0 \implies \int_0^\infty w(t)^\top Qw(t)\ dt = 0$$

which only happens if $w(t) = e^{At}z = 0$ for all $t \geq 0$, from which one concludes that $z = 0$, because e^{At} is non-singular (recall that all state transition matrices are non-singular!). Thus P is positive definite.

4. We argue this by *prove by contradiction.*

   Assume there are matrices $\bar{P}, P$ with $P \neq \bar{P}$ that both solve the Lypunov equation:

$$A^\top P + PA = -Q, \quad \text{and} \quad A^\top \bar{P} + \bar{P}A = -Q$$

   Then

$$A^\top(P - \bar{P}) + (P - \bar{P})A = 0$$

   Multiplying by $e^{A^\top t}$ and e^{At} on the left and right, respectively, we conclude that

$$e^{A^\top t}A^\top(P - \bar{P})e^{At} + e^{A^\top t}(P - \bar{P})Ae^{At} = 0, \quad \forall\ t \geq 0$$

   Yet, it is also the case that

$$\frac{d}{dt}\left(e^{A^\top t}(P - \bar{P})e^{At}\right) = e^{A^\top t}A^\top(P - \bar{P})e^{At} + e^{A^\top t}(P - \bar{P})Ae^{At} = 0$$

   implying that $e^{A^\top t}(P - \bar{P})e^{At}$ must be constant. But,because of stability, this quantity must converge to zero as $t \to \infty$, so it must be always zero. Since e^{At} is nonsingular, this is possible only if $P = \bar{P}$.

The implication that item c $\implies$ item d follows immediately, because if we select $Q = -I$ in condition c, then the matrix P that solves the Lyapunov equation also satisfies $A^\top P + PA < 0$.

Now one way to complete the proof is to show that item d implies item b. Indeed, let P be a symmetric positive-definite matrix for which

$$A^\top P + PA < 0 \tag{6}$$

holds and define

$$Q = -(A^\top P + PA)$$

Consider an arbitrary solution to the LTI system and define the scalar time-dependent map

$$v(t) = x^\top(t)Px(t) \geq 0$$

Taking derivatives, we have

$$\dot{v} = \dot{x}^\top Px + x^\top P\dot{x} = x^\top(A^\top P + PA)x = -x^\top Qx \leq 0$$

Thus, $v(t)$ is nonincreasing and we conclude that

$$v(t) = x^\top(t)Px(t) \leq v(0) = x^\top(0)Px(0)$$

But since $v = x^\top Px \geq \lambda_{\min}(P)\|x\|^2$ we have that

$$\|x\|^2 \leq \frac{x^\top(t)Px(t)}{\lambda_{\min}(P)} = \frac{v(t)}{\lambda_{\min}(P)} \leq \frac{v(0)}{\lambda_{\min}(P)}$$

which means that the system is stable (since the trajectories are bounded). To verify that it is actually exponentially stable, we go back to the derivative of v, and using the facts that $x^\top Qx \geq \lambda_{\min}(Q)\|x\|^2$ and $v = x^\top Px \leq \lambda_{\max}(P)\|x\|^2$, we get that

$$\dot{v} = -x^\top Qx \leq -\lambda_{\min}(Q)\|x\|^2 \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}v, \quad \forall\ t \geq 0$$

To finish we need a comparison lemma. Comparison results are commonly used analysis tools in control theory and hence, it is worth think a bit about them to better understand their utility.

Lemma 24 (Comparison Lemma)**.** Let $v(t)$ be a differentiable scalar signal. For some constant $\mu \in \mathbb{R}$,

$$\dot{v} \leq \mu v(t),\ \forall t \geq t_0 \implies v(t) \leq e^{\mu(t-t_0)} v(t_0),\ \forall t \geq t_0$$

Applying this lemma, we get that

$$v(t) \leq e^{-\lambda(t-t_0)} v(t_0),\ \forall t \geq 0, \quad \lambda = -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$$

which shows that $v(t)$ converges to zero exponentially fast and so does $\|x(t)\|$. □

Analogous Conditions for DT Linear Systems. There are discrete time versions of the above results. Indeed, consider

$$x_{k+1} = Ax_k$$

then we have a similar theorem.

Theorem 25 (DT Lyapunov Stability)**.** The following four conditions are equivalent:

a. The DT LTI system is asymptotically (exponentially) stable.
b. All the eigenvalues of A have magnitude strictly smaller than 1.
c. For every symmetric positive definite Q, there exists a unique solution P to the following Stein equation (aka the discrete-time Lyapunov equation):

$$A^\top PA - P = -Q$$

Moreover, P is symmetric and positive-definite.
d. There exists a symmetric positive-definite matrix P for which the following Lyapunov matrix inequality holds:

$$A^\top PA - P < 0$$

Question: Can you construct an analogous proof?

3.1 The Lyapunov Operator

The Lypunov operator is defined by

$$\mathcal{L}(P) = A^\top P + PA$$

Recall that in [510] you saw that the eigenvalues of this operator are $(\lambda_i + \lambda_j)$ where λ_i, λ_j are eigenvalues of A. Hence, we have the following result.

Proposition 26. $\mathcal{L}$ is non-singular if and only if A and $-A$ share no common eigenvalues.

As a consequence of this proposition, we have the following facts.

Fact. If A is stable, then the Lyapunov operator is non-singular. If A has imaginary eigenvalues, then the Lyapunov operator is singular (since they have to come in complex conjugate pairs). Thus if A is stable, for any Q there is exactly one solution P of Lyapunov equation $A^\top P + PA + Q = 0$.

If A is stable, and P is a unique solution of $A^\top P + PA + Q = 0$, then

$$V(z) = z^\top Pz = z^\top \left(\int_0^\infty e^{tA^\top} Q e^{tA}\, dt \right) z = \int_0^\infty x(t)^\top Q x(t)\, dt$$

The interpretation of this is that $V(z)$ is a cost-to-go from point z given no input.

As in the above proof, if A is stable and $Q > 0$, then for each t

$$e^{tA^\top} Q e^{tA} > 0$$

so that

$$P = \int_0^\infty e^{tA^\top} Q e^{tA} \, dt > 0$$

This means that if A is stable, we can choose any positive definite quadratic form $z^\top Q z$ as the dissipation, i.e. $-\dot{V} = z^\top Q z$. Then, we can solve a set of linear equations to find the unique quadratic function form $V(z) = z^\top P z$. And, V will be positive definite so it is a Lyapunov function that proves A is stable.

Conclusion. That is, a linear system is stable if and only if there is a quadratic Lyapunov function that proves it.

3.2 Other applications of Lyapunov functions

We can use the Lyapunov equation to assess state feedback. This is explored in the Jupyter Notebook on stability for the linearized pendulum system which you can find here on the course website. Consider

$$\dot{x} = Ax + Bu, \ y = Cx, \ u = Kx, \ x(0) = x_0$$

If we apply the state-feedback input $u = Kx$, then we get closed loop dynamics

$$\dot{x} = (A + BK)x$$

Suppose the closed loop dynamics $\dot{x} = (A + BK)x$ are stable. That is, the matrix $A + BK$ is Hurwitz stable meaning its eigenvalues lie in the open left half complex plane.

Then, to evaluate quadratic integral performance measures

$$J_u = \int_0^\infty u(t)^\top u(t) \, dt, \ J_y = \int_0^\infty y(t)^\top y(t) \, dt$$

(which are basically the energy of the input and the energy of the output) we can solve Lyapunov equations

$$(A + BK)^\top P_u + P_u(A + BK) + K^\top K = 0$$

and

$$(A + BK)^\top P_y + P_y(A + BK) + C^\top C = 0$$

so that we have

$$J_u = x_0^\top P_u x_0, \ J_y = x_0^\top P_y x_0$$

Indeed,

$$J_u = \int_0^\infty u(t)^\top u(t) \, dt = \int_0^\infty x(t)^\top K^\top K x(t) \, dt$$

and if we think about this as being our Lyapunov function, that is

$$J_u(t) = z^\top P_u z = \int_0^\infty x(t)^\top K^\top K x(t) \, dt$$

Then, we simply need to solve for P_u using the above Lyapunov equation and then evaluate at $z = x_0$.

References

[1] Shankar Sastry. *Nonlinear systems: analysis, stability, and control*, volume 10. Springer Science & Business Media, 2013.