AA/EE547: Wi 22

Module 1: Introduction to Linear Systems & ODEs

Lecturer: L.J. Ratliff

Throughout the quarter we will use the following keys for references to books:

- **[Ax]**: Axler, Linear Algebra Done Right
- **[C&D]**: Callier and Desoer, Linear Systems Theory
- **[He]**: Hespanha, Linear Systems Theory

References: **Solutions to ODEs**: Chapter 3 **[C&D]**; **Jordan Form**: Chapter 4 **[C&D]**; Chapter 8.D **[Ax]**

1 M1-RL1: Solutions to Autonomous LTI Systems

We will start with continuous time (CT) and then afterwards give the analogous results for discrete time (DT).

Recall from your basic ODE class (or 510) that scalar ODE problems have solutions of the form:

$$\dot{x} = \frac{dx}{dt} = ax(t),\ x(t_0) = x_0 \implies x(t) = x_0 e^{a(t-t_0)}$$

This is easy to check:

a. it satisfies the ODE
$$\frac{d}{dt}(x_0 e^{a(t-t_0)}) = a\underbrace{x_0 e^{a(t-t_0)}}_{x(t)} = ax(t)$$

b. it satisfies the initial condition
$$x(t_0) = x_0 e^{a(t_0-t_0)} = x_0$$

How do we generalize to higher dimensions? Consider a simple mechanical system in Figure 1: spring-mass system. Hooke's Law says that if the spring is stretched (or compressed) z units from its natural length, then it exerts a force that is proportional to z—that is,

$$\text{restoring force} = -kz$$

where k is a positive constant called the spring constant. If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

$$m\ddot{z} = -kz \quad \text{or} \quad m\ddot{z} + kz = 0$$

For an ODE to be "well-posed" we also need sufficient initial data or boundary conditions. In this case, we have a second order ODE so we need two initial conditions:

$$z(0) = z_0,\ z(1) = z_1$$

Two solution approaches:

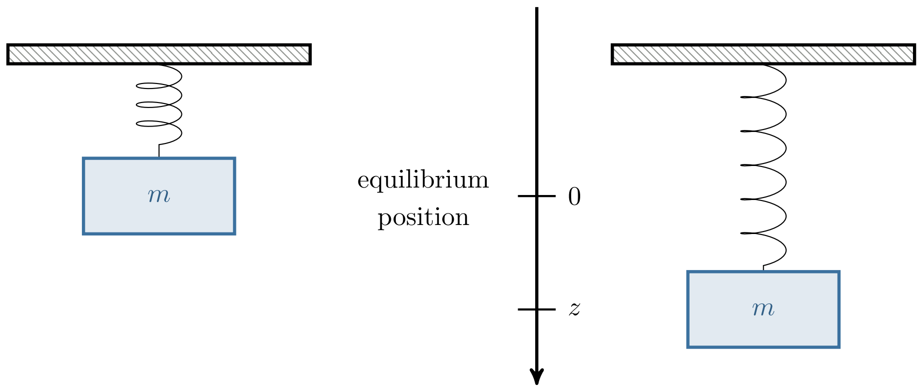


Figure 1: Vertical Pull

1. **Classical**: Guess the solution $z(t) = e^{\lambda t}$, and solve via

$$e^{\lambda t} \implies m\lambda^2 e^{\lambda t} + k e^{\lambda t} = 0 \implies m\lambda^2 + k = 0 \implies \lambda = \pm i\sqrt{\frac{k}{m}}$$

so that

$$z(t) = c_1 e^{-i\sqrt{\frac{k}{m}}t} + c_2 e^{i\sqrt{\frac{k}{m}}t}$$

Then we use the initial data to solve for c_1 and c_2.

2. **State-Space**: First, we rewrite the second order ODE as a system of first order ODEs. Indeed, define $x_1 = z$ and $x_2 = \dot{z}$ so that

$$x_1 = z, x_2 = \dot{z} \implies \dot{x}_1 = \dot{z} = x_2, \text{ and } \dot{x}_2 = \ddot{z} = -\frac{k}{m}z = -\frac{k}{m}x_1$$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{k}{m}x_1 \end{aligned}\right\} \iff \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \iff \dot{x} = Ax$$

where $x \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$. The solution is then given by

$$x(t) = e^{At}x_0,\ x(t_0) = x_0$$

where e^{At} is the matrix exponential.

Aside: There are many ways to compute the matrix exponential:

- Laplace transform

$$e^{At} = \mathcal{L}^{-1}((sI - A)^{-1})$$

- Series expansion with Cayley-Hamilton

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

[510] Cayley-Hamilton: A matrix $A \in \mathbb{R}^{n\times n}$ satisfies its own characteristic polynomial $p_A(A) = 0$ so that A^n can be expressed as a linear combination of lower powers of A. This approach works well with nilpotent matrices—i.e., where some power of A is identical to zero.

- Similarity transform to Jordan form

$$e^{At} = P^{-1}e^{Jt}P$$

For this system, we use the third way because this system is such that A is "diagonalizable". The eigenpairs of A are

$$\{(\lambda, v)\} = \left\{ \left(-i\sqrt{\frac{k}{m}}, \begin{bmatrix} i\sqrt{\frac{k}{m}} \\ -i\sqrt{\frac{k}{m}} \end{bmatrix} \right), \left(i\sqrt{\frac{k}{m}}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right\}$$

Hence,

$$e^{At} = P^{-1}e^{\Lambda t}P = \begin{bmatrix} -i\sqrt{\frac{k}{m}} & 1 \\ i\sqrt{\frac{k}{m}} & 1 \end{bmatrix}^{-1} \begin{bmatrix} e^{-i\sqrt{\frac{k}{m}}t} & 0 \\ 0 & e^{i\sqrt{\frac{k}{m}}t} \end{bmatrix} \begin{bmatrix} -i\sqrt{\frac{k}{m}} & 1 \\ i\sqrt{\frac{k}{m}} & 1 \end{bmatrix} = \begin{bmatrix} \cos\left(\sqrt{\frac{5}{2}}t\right) & \sqrt{\frac{2}{5}}\sin\left(\sqrt{\frac{5}{2}}t\right) \\ -\sqrt{\frac{2}{5}}\sin\left(\sqrt{\frac{5}{2}}t\right) & \cos\left(\sqrt{\frac{5}{2}}t\right) \end{bmatrix}$$

and

$$x(t) = \begin{bmatrix} \cos\left(\sqrt{\frac{5}{2}}t\right) & \sqrt{\frac{2}{5}}\sin\left(\sqrt{\frac{5}{2}}t\right) \\ -\sqrt{\frac{2}{5}}\sin\left(\sqrt{\frac{5}{2}}t\right) & \cos\left(\sqrt{\frac{5}{2}}t\right) \end{bmatrix} \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix}$$

1.1 General CT LTI Systems with Inputs

Consider now the general LTI system in state-space form:

$$\dot{x} = Ax + Bu \tag{1}$$
$$y = Cx + Du \tag{2}$$

where

- $x \in \mathbb{R}^n$ is the "state" of the system
- $u \in \mathbb{R}^m$ is the "input" to the system
- $y \in \mathbb{R}^p$ is the "output" of the system
- $A \in \mathbb{R}^{n \times n}$ describes how the state changes in time (dynamics)
- $B \in \mathbb{R}^{n \times m}$ describes how the input effects the state dynamics
- $C \in \mathbb{R}^{p \times n}$ describes how the state is transformed to the output
- $D \in \mathbb{R}^{p \times m}$ describes how the input is transformed to the output (for the most part in this class we take $D = 0$).

The solution to the CT LTI system in (1) is

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\, d\tau$$

Let's check!

1. Satisfies the ODE:

$$\begin{aligned} \dot{x}(t) &= \frac{d}{dt}e^{A(t-t_0)}x_0 + \frac{d}{dt}\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\, d\tau \\ &= Ae^{A(t-t_0)}x_0 + e^{A(t-t)}Bu(t) + \int_{t_0}^{t} Ae^{A(t-\tau)}Bu(\tau)\, d\tau \\ &= A\,\underbrace{e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\, d\tau}_{x(t)} + Bu(t) \end{aligned}$$

2. Satisfies the initial condition:

$$x(t_0) = e^{A(t_0-t_0)}x_0 + \cancelto{0}{\int_{t_0}^{t_0} e^{A(t_0-\tau)}Bu(\tau)\, d\tau} = x_0$$

Leibniz Rule:

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(t,\tau)d\tau\right) = f(t,b(t))\frac{d}{dt}b(t) - f(t,a(t))\frac{d}{dt}a(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}f(t,\tau)d\tau$$

Review: If you need a reminder of how to compute the matrix expoential, I have posted the [510] lectures notes for this in `lec0-510Review.pdf`

1.2 DT LTI Systems

A discrete time LTI system is given by

$$x[k+1] = Ax[k] + Bu[k] \tag{3}$$
$$y[k] = Cx[k] + Du[k] \tag{4}$$

The solution for the DT LTI system is given by

$$x[k] = A^{k-k_0}x[k_0] + \sum_{\ell=k_0}^{k-1} A^{k-\ell-1}Bu[\ell]$$

One can easily check this expression for $x[k]$ satisfies the recursion:

- It holds for the initial point $k = k_0$:

$$x[k_0+1] = A^{k_0+1-k_0}x[k_0] + \sum_{\ell=k_0}^{k_0+1-1} A^{k_0+1-\ell-1}Bu[\ell] = Ax[k_0] + Bu[k_0]$$

- It holds for arbitrary k since

$$\begin{aligned}
x[k+1] &= A^{k+1-k_0}x[k_0] + \sum_{\ell=k_0}^{k+1-1} A^{k+1-\ell-1}Bu[\ell] \\
&= AA^{k-k_0}x[k_0] + A\left(\sum_{\ell=k_0}^{k} A^{k-\ell-1}Bu[\ell]\right) \\
&= AA^{k-k_0}x[k_0] + A\left(\sum_{\ell=k_0}^{k-1} A^{k-\ell-1}Bu[\ell] + A^{-1}Bu[k]\right) \\
&= A\underbrace{\left(A^{k-k_0}x[k_0] + \sum_{\ell=k_0}^{k-1} A^{k-\ell-1}Bu[\ell]\right)}_{x[k]} + Bu[k]
\end{aligned}$$

Hence, the solution to the DT LTI system in (3) is

$$x[k] = A^{k-k_0}x[k_0] + \sum_{\ell=k_0}^{k-1} A^{k-\ell-1}Bu[\ell]$$

2 M1-RL2: LTV Systems & Fundamental Theorem for ODEs

Consider the general ODE

$$\dot{x} = f(x,t)$$

The function f must satisfy two assumptions:

(A1) Let $\mathcal{D}$ be a set in $\mathbb{R}_+$ which contains at most a finite number of points per unit interval. $\mathcal{D}$ is the set of possible discontinuity points; it may be empty. For each fixed, $x \in \mathbb{R}^n$, the function $t \in \mathbb{R}_+\backslash\mathcal{D} \to f(x,t) \in \mathbb{R}^n$ is continuous and for any $\tau \in \mathcal{D}$, the left-hand and right-hand limits $f(x,\tau_-)$ and $f(x,\tau_+)$, respectively, are finite vectors in $\mathbb{R}^n$.

(A2) There is a piecewise continuous function $k(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(\xi,t) - f(\xi',t)\| \leq k(t)\|\xi - \xi'\| \quad \forall\, t \in \mathbb{R}_+,\ \forall \xi, \xi' \in \mathbb{R}^n$$

This is called a global Lipschitz condition because it must hold for all ξ and ξ'.

Theorem. (Fundamental Theorem of Existence and Uniqueness.) Consider

$$\dot{x}(t) = f(x,t)$$

where initial condition (t_0, x_0) is such that $x(t_0) = x_0$. Suppose f satisfies (A1) and (A2). Then,

1. For each $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exists a continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}^n$ such that

$$\phi(t_0) = x_0$$

and

$$\dot{\phi}(t) = f(\phi(t), t), \quad \forall t \in \mathbb{R}_+\backslash\mathcal{D}$$

2. This function is unique. The function ϕ is called the solution through (t_0, x_0) of the differential equation.

2.1 Applying it to Linear Time Varying Systems

Recall

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \quad \text{(state DE)} \\ y(t) &= C(t)x(t) + D(t)u(t) \quad \text{(read-out eqn.)} \end{aligned}$$

with initial data (t_0, x_0) and the assumptions on $A(\cdot), B(\cdot), C(\cdot), D(\cdot), u(\cdot)$ all being PC:

- $A(t) \in \mathbb{R}^{n\times n}$
- $B(t) \in \mathbb{R}^{n\times m}$
- $C(t) \in \mathbb{R}^{p\times n}$
- $D(t) \in \mathbb{R}^{p\times m}$

The input function $u(\cdot) \in \mathcal{U}$, where $\mathcal{U}$ is the set of piecewise continuous functions from $\mathbb{R}_+ \to \mathbb{R}^m$.

This system satisfies the assumptions of our existence and uniqueness theorem. Indeed,

1. For all fixed $x \in \mathbb{R}^n$, the function $t \in \mathbb{R}_+ \backslash \mathcal{D} \to f(x,t) \in \mathbb{R}^n$ is continuous where $\mathcal{D}$ contains all the points of discontinuity of $A(\cdot), B(\cdot), C(\cdot), D(\cdot), u(\cdot)$
2. There is a PC function $k(\cdot) = \|A(\cdot)\|$ such that

$$\|f(\xi,t) - f(\xi',t)\| = \|A(t)(\xi - \xi')\| \leq k(t)\|\xi - \xi'\| \;\; \forall t \in \mathbb{R}_+, \;\forall \xi, \xi' \in \mathbb{R}^n$$

Hence, by the above theorem, the differential equation has a unique continuous solution $x : \mathbb{R}_+ \to \mathbb{R}^n$ which is clearly defined by the parameters $(t_0, x_0, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times U$.

Theorem. (Existence of the state transition map/flow.) Under the assumptions and notation above, for every triple $(t_0, x_0, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times U$, the state transition map

$$x(\cdot) = \phi(\cdot, t_0, x_0, u) : \mathbb{R}_+ \to \mathbb{R}^n$$

is a continuous map well-defined as the unique solution of the state differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

with (t_0, x_0) such that $x(t_0) = x_0$ and $u(\cdot) \in U$.

2.2 Zero-State and Zero-Input Maps

The state transition function of a linear system is equal to its zero-input state transition function and its zero-state state transition map:

$$\phi(t, t_0, x_0, u) \;\; = \;\; \underbrace{\phi(t, t_0, x_0, 0)}_{\text{zero-input state trans. func.}} + \underbrace{\phi(t, t_0, 0, u)}_{\text{zero-state state trans. func.}}$$

Due to the fact that the state transition map and the response map are linear, they have the property that for fixed $(t, t_0) \in \mathbb{R}_+ \times \mathbb{R}_+$ the maps

$$\phi(t, t_0, \cdot, 0) : \mathbb{R}^n \to \mathbb{R}^n : x_0 \mapsto \phi(t, t_0, x_0, 0)$$

Hence by the *Matrix Representation Theorem*, they are representable by matrices. Therefore there exists a matrix $\Phi(t, t_0) \in \mathbb{R}^{n \times n}$ such that

$$\phi(t, t_0, x_0, 0) = \Phi(t, t_0)x_0, \quad \forall x_0 \in \mathbb{R}^n$$

(State transition matrix.) $\Phi(t, t_0)$ is called the state transition matrix.

Consider the matrix differential equation

$$\dot{X} = A(t)X, \quad X(\cdot) \in \mathbb{R}^{n \times n}$$

Let $X(t_0) = X_0$.

The state transition matrix $\Phi(t, t_0)$ is defined to be the solution of the above matrix differential equation starting from $\Phi(t_0, t_0) = I$. That is,

$$\frac{\partial}{\partial t}\Phi(t, t_0) = A(t)\Phi(t, t_0)$$

and $\Phi(t_0, t_0) = I$.

2.3 State Transition Matrix

Definition. (LTI State transition matrix.) The state transition matrix for

$$\dot{x} = Ax,\ x \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n}$$

is the *matrix exponential* e^{At} defined to be

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots$$

where I is the $n \times n$ identity matrix.

Proof. It is easy to verify that

$$\Phi(t,0) = e^{At} \quad \text{and} \quad \Phi(t,t_0) = e^{A(t-t_0)}$$

by checking that

$$x(t) = e^{A(t-t_0)}x_0$$

satisfies the differential equation

$$\dot{x} = Ax, \quad x(t_0) = x_0$$

Indeed, by the fact that $\Phi(t,t_0)$ satisfies the ODE (by definition) we know that

$$\frac{\partial}{\partial t}\Phi(t,t_0) = A\Phi(t,t_0)$$

and

$$\frac{\partial}{\partial t}\exp(A(t-t_0))x_0 = A\exp(A(t-t_0))x_0$$

In addition, $\Phi(t_0,t_0) = I$ and $\exp(A(t_0-t_0)) = I$. Hence, $\Phi(t,t_0)$ and $\exp(A(t-t_0))$ satisfy the same ODE so they are equal. □

2.4 Properties of State Transition Function

The state transition matrix satisfies several important properties whicih we will use throughout to make arguments about solutions to linear differential equations.

Proposition 1. The following properties hold:

1. The solution of $\dot{x} = A(t)x$, $\phi(t,t_0,x_0,0)$ is given by

$$\phi(t,t_0,x_0,0) = \Phi(t,t_0)x_0 \tag{5}$$

2. For all $t,t_0,t_1 \in \mathbb{R}_+$,

$$\Phi(t,t_0) = \Phi(t,t_1)\Phi(t_1,t_0)$$

3. The inverse of the state transition matrix is

$$(\Phi(t,t_0))^{-1} = \Phi(t_0,t)$$

4. The determinant is give by

$$\det\,\Phi(t,t_0) = \exp\left(\int_{t_0}^{t}\text{trace}(A(\tau))\,d\tau\right)$$

Proof. Call the left-hand side of (5) LHS, and the right-hand side RHS.

1. Check first that the LHS of (5) and the RHS are equal at t_0:

$$\text{LHS}(t_0) = \phi(t_0, t_0, x_0) = x_0 \quad \text{and} \quad \text{RHS}(t_0) = \Phi(t_0, t_0)x_0 = Ix_0 = x_0$$

   Now, we check they satisfy the same differential equation:

$$\frac{d}{dt}\text{LHS}(t) = A(t)\text{LHS}(t) \quad \text{and} \quad \frac{d}{dt}\text{RHS}(t) = A(t)\text{RHS}(t)$$

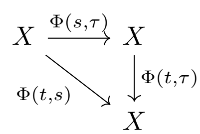
   so that $\phi(t, t_0, x_0) = \Phi(t, t_0)x_0$.

2. Again we use the same trick of checking the initial condition and the differential equation (and invoke the existence and uniqueness theorem).

$$\begin{aligned} \text{RHS}(t_1) &= \text{LHS}(t_1) \\ \frac{d}{dt}\text{RHS}(t) &= A(t)\text{RHS}(t) \\ \frac{d}{dt}\text{LHS}(t) &= A(t)\text{LHS}(t) \end{aligned}$$

   Hence, LHS $\equiv$ RHS.

3. First, $\Phi(t, s) = \Phi(s, \tau)\Phi(\tau, t)$ for any t, s, τ since the following diagram commutes:



   Indeed, consider the unique solution to

$$\begin{cases} \dot{x}(\sigma) &= A(\sigma)x(\sigma) \\ x(s) &= a \end{cases}$$

   Then, $x(t) = \Phi(t, s)a$, $x(t) = \Phi(t, \tau)x(\tau)$ and $x(\tau) = \Phi(\tau, s)a$, and hence

$$\Phi(t, \tau)\Phi(\tau, s)a = \Phi(t, s)a$$

   that is

$$(\Phi(t, \tau)\Phi(\tau, s) - \Phi(t, s))a = 0$$

   since this must hold for all $a \in \mathbb{R}^n$, the claim holds.

   We claim that $\Phi(t, s)$ is invertible and that its inverse is given by $\Phi(s, t)$. Indeed, from $\Phi(t, s) = \Phi(s, \tau)\Phi(\tau, t)$ we have that

$$I = \Phi(t, s)\Phi(s, t)$$

   Thus, $\Phi(t, t_0)$ is invertible for all t. Hence,

$$\Phi(t_0, t_0) = I = \Phi(t_0, t)\Phi(t, t_0) \implies \Phi(t, t_0)^{-1} = \Phi(t_0, t)$$

4. This is called the *Jacobi-Liouville* equation. We will take this one as given.

□

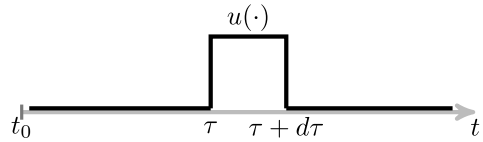


Figure 2: Input

2.5 Solving Linear ODE via $\Phi(t, t_0)$

Heuristic Derivation. First, let us consider a *heuristic* derivation of the zero-state transition. (page 35 of **[C&D]**) Consider the input in Fig. 2.

Then,

$$x(\tau) = \Phi(\tau, t_0)x_0$$

and

$$x(\tau + d\tau) = x(\tau) + [A(\tau)x(\tau) + B(\tau)u(\tau)]d\tau$$

$$\begin{aligned} x(t) &= \Phi(t, \tau + d\tau)x(\tau + d\tau) \\ &= \Phi(t, \tau + d\tau)[x(\tau) + A(\tau)x(\tau)d\tau + B(\tau)u(\tau)d\tau] \\ &= \Phi(t, \tau + d\tau)[I + A(\tau)d\tau]x(\tau) + \Phi(t, \tau + d\tau)B(\tau)u(\tau)d\tau \\ &\simeq \Phi(t, \tau + d\tau)\Phi(\tau + d\tau, \tau)\Phi(\tau, t_0)x_0 + \Phi(t, \tau + d\tau)B(\tau)u(\tau)d\tau \end{aligned}$$

Hence,

$$x(t) \simeq \Phi(t, t_0)x_0 + \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

Simple scalar autonomous example. Consider the case where $n = 1$ so that we have

$$\dot{x}(t) = a(t)x(t), \ x(0) = x_0, \ a(t) \in \mathbb{R}$$

Suppose that $x(t) \neq 0$ for all t. Then, we can find the solution by integration. That is

$$\int_0^t a(\tau)\ d\tau = \int_0^t \frac{\dot{x}(\tau)}{x(\tau)}\ d\tau = \log(x(t)) - \log(x(0))$$

Solving for $x(t)$, we have

$$x(t) = x_0 \exp\left(\int_0^t a(\tau)\ d\tau\right)$$

For instance, if we have

$$\dot{x}(t) = -tx(t), \ x(0) = 2$$

Then, the solution is

$$x(t) = e^{-\int_0^t \tau\ d\tau} = 2e^{-\frac{1}{2}t^2}$$

Simple scalar autonomous example. What if we have an input? Consider again the scalar case where $n = 1$ and $m = 1$ with dynamics

$$\dot{x}(t) = a(t)x(t) + b(t), \quad x(0) = x_0$$

Then the unique solution is given by

$$x(t) = \exp\left(\int_0^t a(\tau)\ d\tau\right)x_0 + \int_0^t \exp\left(\int_\tau^t a(s)\ ds\right)b(\tau)\ d\tau$$

2.5.1 Solution for LTV System More Generally

Theorem 2 (Solution of Linear System)**.** The solution to

$$\dot{x} = A(t)x(t) + B(t)u(t), \; x(t_0) = x_0 \tag{6}$$

is given by

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau) \; d\tau \tag{7}$$

Proof. Proof idea: We will use the trick that checks the equality by showing the left and right hand sides of (7) satisfy the same ODE. That is, at t_0, they have the same value (initial condition) and the derivative of the left and right hand sides is the same. The key here is that since we have the existence and uniqueness theorem, we know then that the solution of the ODE is unique, so that means any two expressions that satisfy it have to be equal.

Towards this end, let

$$y(t) := \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau) \; d\tau$$

be the right-hand side of (7), and $x(t)$ be the left hand side of (7). Then we simply need to check that

1. $x(t)$ and $y(t)$ satisfy the same differential equation
2. $x(t_0)$ and $y(t_0)$ has the same value (i.e., same initial condition).

1. Verifying $y(t)$ and $x(t)$ satisfy the same ODE. The mapping $x(t)$ trivially satisfies (6), by definition. That is, (6) states

$$\dot{x} = A(t)x(t) + B(t)u(t)$$

So, let's check that $y(t)$ does. Indeed, recall that $\Phi(t, t_0)$ is a state transition matrix meaning it is a solution to $\dot{X} = A(t)X$—i.e.,

$$\frac{\partial}{\partial t}\Phi(t, t_0) = A(t)\Phi(t, t_0)$$

Hence, we have

$$\begin{aligned}
\dot{y} := \frac{d}{dt}y(t) &= \frac{d}{dt}\Phi(t, t_0)x_0 + \frac{d}{dt}\int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau) \; d\tau \\
&= A(t)\Phi(t, t_0)x_0 + \frac{d}{dt}(t)\Phi(t, t)B(t)u(t) - \underbrace{\frac{d}{dt}(t_0)}_{\to 0}\Phi(t, t_0)B(t_0)u(t_0) + \int_{t_0}^{t} \frac{d}{dt}\left(\Phi(t, \tau)B(\tau)u(\tau)\right) d\tau \\
&= A(t)\Phi(t, t_0)x_0 + B(t)u(t) + A(t)\int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau) \; d\tau \\
&= A(t)y(t) + B(t)u(t)
\end{aligned}$$

That completes the proof of 1.

2. Verifying the satisfy the same initial condition. Again, $x(t_0) = x_0$ is given in the problem definition (6). Now, for $y(t_0)$ we have

$$y(t_0) = \underbrace{\Phi(t_0, t_0)}_{I}x_0 + \underbrace{\int_{t_0}^{t_0} \Phi(t, \tau)B(\tau)u(\tau) \; d\tau}_{\to 0} = x_0$$

which completes the proof of 2. $\square$

Thus, we have that the state transition function is given by

$$\boxed{\phi(t, t_0, x_0, u_{[t_0,t]}) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)\ d\tau}$$

The solution above satisfies two axioms:

1. **state transition axiom**: $\forall x_0 \in \mathbb{R}^n$, if $u, v \in \mathcal{U}$ such that $u \equiv v$ on $t \in [t_0, t_1]$, then

$$x(t_1) = \phi(t_1, t_0, x_0, u) = \phi(t_1, t_0, x_0, v)$$

   This means that the state at time $t_1 \geq t_0$—namely, $x(t_1)$—depends only on the values of $u \in \mathcal{U}$ in the elapsed interval which is why we write $u_{[t_0,t_1]}$ in u argument of ϕ.
2. **state composition axiom**:

$$x(t_2) = \phi(t_2, t_0, x_0, u) = \phi(t_2, t_1, \phi(t_1, t_0, x_0, u), u), \quad t_0 \leq t_1 \leq t_2 \in \mathbb{R}_+$$

   Indeed, we have that

$$\begin{aligned}
\phi(t_2, t_1, \phi(t_1, t_0, x_0, u_{[t_0,t_1]}), u_{[t_1,t_2]}) &= \Phi(t_2, t_1)\left(\Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\ d\tau\right) \\
&\quad + \int_{t_1}^{t_2} \Phi(t_2, \tau)B(\tau)u(\tau)\ d\tau \\
&= \Phi(t_2, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_2, \tau)B(\tau)u(\tau)\ d\tau + \int_{t_1}^{t_2} \Phi(t_2, \tau)B(\tau)u(\tau)\ d\tau \\
&= \Phi(t, t_0)x_0 + \int_{t_0}^{t_2} \Phi(t_2, \tau)B(\tau)u(\tau)\ d\tau \\
&= \phi(t_2, t_0, x_0, u_{[t_0,t_2]})
\end{aligned}$$

2.6 Discrete Time LTV Systems

Consider

$$\begin{aligned}
x_{k+1} &= A_k x_k + B_k u_k \\
y_k &= C_k x_k + D_k u_k
\end{aligned}$$

where $A_k \equiv A(k)$ and similarly for the other system matrices, states and control inputs.

While I will not cover this in detail (please see Chapter 2d **[C&D]**), we note that there is an equivalent "fundamental theorem of ODEs" for linear difference equations and we can define completely analogous state transition functions and response maps which are linear and hence, have corresponding matrix representations. Indeed, the state transition matrix $\Phi(\cdot,\cdot) : \mathbb{N}_+ \times \mathbb{N}_+ \to \mathbb{R}^{n\times n}$ is defined as follows:

$$\forall\ k_0,\ \ \Phi(k+1, k_0) = A_k\Phi(k, k_0),\ \ k = k_0, k_0 + 1, \dots,\ \ \Phi(k_0, k_0) = I$$

Note: It is only when A_k is non-singular for all $k \in \mathbb{N}$, that $\Phi(k + 1, k_0) = A_k\Phi(k, k_0)$ can be solved for $\Phi(k, k_0)$ in terms of $\Phi(k + 1, k_0)$.

It turns out that

$$\Phi(k, k_0) = A_{k-1}A_{k-2}\cdots A_{k_0}$$

Practice Problem. Show this by induction.

The discrete time state transition matrix also has analogous properties as those for its continuous time counter part. Indeed, we have the following composition property:

$$\Phi(k, k_0) = \Phi(k, k_1)\Phi(k_1, k_0),\ \forall k_0 \le k_1 \le k$$

The solution and output can then be characterized by

$$\begin{aligned} x_k &= \Phi(k, k_0)x_0 + \sum_{\ell=k_0}^{k-1} \Phi(k, \ell+1)B_\ell u_\ell \\ y_k &= C_k\Phi(k, k_0)x_0 + C_k\left(\sum_{\ell=k_0}^{k-1} \Phi(k, \ell+1)B_\ell u_\ell\right) + D_k u_k \end{aligned}$$

Appendix

This appendix contains additional mathematical preliminaries that hopefully you have seen in another class, but if not its here as a reminder.

A A brief review of analysis

Definition 3 (Continuity)**.** Let $W \subset \mathbb{R}$ denote an open interval and consider a function $f : W \to \mathbb{R}$. The function f is said to be continuous at the point $t_0 \in U$ if

$$\lim_{t \to t_0} f(t) = f(t_0)$$

exists. That is, for every $\varepsilon > 0$, there exists $\delta(t_0, \varepsilon) > 0$ such that

$$|f(t) - f(t_0)| < \varepsilon$$

whenever $|t - t_0| < \delta(t_0, \varepsilon)$ and $t \in W$. If f is continuous at every point in W, then we just say f is a continuous function or we say f is continuous on W.

Note that δ depends on the choice of t_0 and ε in the above definition. If at each $t_0 \in W$, it is true that there is a $\delta > 0$ independent of t_0 (i.e., $\delta = \delta(\varepsilon)$), such that $|f(t) - f(t_0)| < \varepsilon$ whenever $|t - t_0| < \delta$ and $t \in W$, then we say f is *uniformly continuous* on W.

We denote by $C(W, \mathbb{R})$ the space of real-valued continuous functions on W.

Definition 4. Piecewise Continuity. A function or curve is piecewise continuous if it is continuous on all but a finite number of points in any compact (closed and bounded in $\mathbb{R}$) interval.

For a function $f : W \to \mathbb{R} : t \mapsto z$, let $f^{(0)} \equiv f$ and $f^{(k)} \equiv \partial^k f/\partial t^k$. For $W \subset \mathbb{R}$ an open set, given $r \in \mathbb{N}$ and W an open set we will use the notation

$$C^r(W, \mathbb{R}) = \{f : W \to \mathbb{R}|\ f^{(j)} \text{ exists on } W \text{ and } f^{(j)} \in C(W, \mathbb{R}),\ j = 0, 1, \ldots, r\}$$

for the C^r-functions. For $W \subset \mathbb{R}^n$ with non-empty interior, we can define $C(W, \mathbb{R})$ and $C^r(W, \mathbb{R})$ similarly. Indeed,

$$C^r(W, \mathbb{R}) = \left\{f : W \to \mathbb{R}|\ f^{(j)} \text{ exists on } W \text{ and } f^{(j)} = \frac{\partial^j f}{(\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \in C(W, \mathbb{R}), \sum_\ell i_\ell = j,\ j = 0, 1, \ldots, r\right\}$$

Let W be a subset of $\mathbb{R}^n$ with non-empty interior and let $f : W \to \mathbb{R}^m$. Then $f = (f_1, \ldots, f_m)^\top$ where $f_i : W \to \mathbb{R}$. We say that $f \in C(W, \mathbb{R}^m)$ if $f_i \in C(W, \mathbb{R})$ for each i and that for some r, $f \in C^r(W, \mathbb{R}^m)$ if $f_i \in C^r(W, \mathbb{R})$ for each i.

A.1 Lipschitz Continuity

Often times it is easy to characterize whether or not a given function satisfies (A2) by checking an auxiliary condition.

Proposition 5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and differentiable. Then

$$f \text{ Lipschitz } \iff \exists K, \text{ s.t. } \forall x \in \mathbb{R}, \ |f'(x)| \leq K$$

To prove this, we need the mean value theorem.

Theorem 6. Mean Value Theorem. If a function f is continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b), then there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. ($\Longleftarrow$). Suppose the derivative is bounded by some K. By the mean value theorem, we have that for $x, y \in \mathbb{R}$, there exists $c \in \mathbb{R}$ such that

$$f(x) - f(y) = (x - y)f'(c)$$

so that

$$|f(x) - f(y)| = |(x - y)f'(c)| \leq K|x - y|$$

Hence, f is Lipschtiz.

($\Longrightarrow$). Suppose f is K-Lipschitz so that $|f(x) - f(y)| \leq K|x - y|$ for all x, y and hence, in particular, $|f(x + h) - f(x)| \leq K|h|$ for all x and h. Then, taking the limit we have that

$$f'(x) = \lim_{h \to 0} \left| \frac{f(x + h) - f(x)}{h} \right| \leq K$$

Alternative argument:

$$f \text{ Lipschitz } \implies |f(x) - f(y)| \leq K|x - y|, \forall x, y \implies \lim_{x \to y} \left| \frac{f(x) - f(y)}{x - y} \right| \leq K$$

□

Note: Lipschitz functions do not have to be differentiable. They have to be almost everywhere differentiable (except on a set of measure zero).

Proposition 7. Given $R > 0$, if there is a PC function $k(\cdot)$ s.t.

$$\|D_1 f(x, t)\| \leq k(t), \quad \forall x \in B_R(0), \ \forall t \in \mathbb{R}_+$$

then the Lipschitz condition in (A2) holds for all $\xi, \xi' \in B_R(0)$, $t \in \mathbb{R}_+$

Examples.

1. The function

   $$f(x) = \sqrt{x^2 + 1}$$

   defined for all real numbers is Lipschitz continuous with the Lipschitz constant $K = 1$, because it is everywhere differentiable and the absolute value of the derivative is bounded above by 1. Indeed,

   $$f'(x) = x(x^2 + 1)^{-1/2}$$

so that

$$|f'(x)| = |x(x^2+1)^{-1/2}| \leq |x||(x^2+1)^{-1/2}|$$

Claim:

$$\frac{|x|}{|(x^2+1)^{1/2}|} \leq 1$$

This is true because

$$|x| = |(x^2)^{1/2}| \leq |(x^2+1)^{1/2}|$$

2. The functions $\sin(x)$ and $\cos(x)$ are Lipschitz with constant $K = 1$ since their derivatives are bounded by 1.
3. Practice. is the function $\sin(x^2)$? what about $\sqrt{x}$? (hint: consider the derivatives)

B Proof of Fundamental Theorem of ODEs

Theorem 8 (Fundamental Theorem of Existence and Uniqueness.)**.** Consider

$$\dot{x}(t) = f(x,t)$$

where initial condition (t_0, x_0) is such that $x(t_0) = x_0$. Suppose f satisfies (A1) and (A2). Then,

a. For each $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exists a continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}^n$ such that

$$\phi(t_0) = x_0$$

and

$$\dot{\phi}(t) = f(\phi(t), t), \quad \forall t \in \mathbb{R}_+ \backslash \mathcal{D}$$

b. This function is unique. The function ϕ is called the solution through (t_0, x_0) of the differential equation.

Note that if the Lipschitz condition does not hold, it may be that the solution cannot be continued beyond a certain time. e.g., consider

$$\dot{\xi}(t) = \xi^2(t), \quad \xi(0) = \frac{1}{c}, \ c \neq 0$$

where $\xi : \mathbb{R}_+ \to \mathbb{R}$. This differential equation has the solution

$$\xi(t) = \frac{1}{c-t}$$

on $t \in (-\infty, c)$. As $t \to c$, $\|\xi(t)\| \to \infty$. This is called finite escape time at c.

We need the following notion of a Cauchy sequence—this is a stronger notion of convergence.

Definition 9 (Cauchy sequence.)**.** A sequence $(v_i)_{i=1}^{\infty}$ in $(V, F, \|\cdot\|)$ is said to be a Cauchy sequence in V if and only if for any $\varepsilon > 0$, $\exists\ N_\varepsilon \in \mathbb{N}$ such that for any pair $m, n > N_\varepsilon$,

$$\|v_m - v_n\| < \varepsilon \ \ \forall p \in \mathbb{N}$$

And, we need the notion of a Banach space.

Definition 10. Banach Space. A Banach space X is a normed linear space that is complete with respect to that norm—that is, every Cauchy sequence $\{x_n\}$ in X converges in X.

Proof sketch for existence. Construct a sequence of continuous functions

$$x_{m+1}(t) = x_0 + \int_{t_0}^{t} f(x_m(\tau), \tau)\ d\tau$$

where $x_0(t_0) = x_0$ and $m = 0, 1, 2, \ldots$. The idea is to show that the sequence of continuous functions $\{x_m(\cdot)\}_0^\infty$ converges to (i) a continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}^n$ which is (ii) a solution of $\dot{x} = f(x, t)$, $x(t_0) = x_0$.

To show (i), we show that $\{x_m(\cdot)\}_0^\infty$ is a Cauchy sequence in a Banach space $(C([t_1, t_2], \mathbb{R}^n), \mathbb{R}, \|\cdot\|_\infty)$, where $t_0 \in [t_1, t_2]$.

To show [(ii)—i.e. that $\phi(\cdot)$ is a solution of the differential equation—recall that

$$x_{m+1}(t) = x_0 + \int_{t_0}^{t} f(x_m(\tau), \tau)\ d\tau$$

By the above argument, $m \to \infty$, $x_m(\cdot) \to \phi(\cdot)$ on $[t_1, t_2]$. Hence, it suffices to show that

$$\int_{t_0}^{t} f(x_m(\tau), \tau)\ d\tau \to \int_{t_0}^{t} f(\phi(\tau), \tau)\ d\tau, \quad \text{as } m \to \infty$$

□

To prove uniqueness, we need the so called Bellman-Gronwall Lemma.

Lemma 11 (Bellman-Gronwall)**.** Let $u(\cdot)$, $k(\cdot)$ be real-valued, piecewise continuous functions on $\mathbb{R}_+$ and assume $u(\cdot), k(\cdot) > 0$ on $\mathbb{R}_+$. Suppose $c_1 > 0$, $t_0 \in \mathbb{R}_+$. If

$$u(t) \leq c_1 + \int_{t_0}^{t} k(\tau) u(\tau)\ d\tau$$

then

$$u(t) \leq c_1 \exp\left(\int_{t_0}^{t} k(\tau)\ d\tau\right)$$

Proof. Without loss of generality], assume $t > t_0$. Let $U(t) = c_1 + \int_{t_0}^{t} k(\tau) u(\tau)\ d\tau$. Thus,

$$u(t) \leq U(t)$$

Multiply both sides of

$$u(t) \leq c_1 + \int_{t_0}^{t} k(\tau) u(\tau)\ d\tau$$

by the non-negative function

$$k(t) \exp\left(-\int_{t_0}^{t} k(\tau)\ d\tau\right)$$

resulting in

$$\frac{d}{dt}\left(U(t) \exp\left(-\int_{t_0}^{t} k(\tau)\ d\tau\right)\right) \leq 0$$

and thus

$$u(t) \leq U(t) \leq c_1 \exp\left(-\int_{t_0}^{t} k(\tau)\ d\tau\right)$$

□

Proof of uniqueness sketch: Invoke Bellman-Grownwall.

Let's consider an example.

Example. Consider

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(t_0) &= x_0 \end{aligned}$$

Show the solution is unique.

Proof. Assume $\phi(t), \psi(t)$ are two solutions so that $\phi(t_0) = \psi(t_0) = x_0$ and

$$\begin{aligned} \dot{\phi}(t) &= A(t)\phi(t) + B(t)u(t) \\ \dot{\psi}(t) &= A(t)\psi(t) + B(t)u(t) \end{aligned}$$

Then

$$\phi(t) - \psi(t) = \int_{t_0}^{t} (A(\tau)\phi(\tau) - A(\tau)\psi(\tau)) \, d\tau$$

so that

$$\|\phi(t) - \psi(t)\| \leq \|A(t)\|_{\infty,[t_0,t]} \int_{t_0}^{t} \|\phi(\tau) - \psi(\tau)\| \, d\tau$$

By Bellman-Gronwall,

$$\|\phi(t) - \psi(t)\| \leq c_1 + \|A(t)\|_{\infty,[t_0,t]} \int_{t_0}^{t} \|\phi(\tau) - \psi(\tau)\| \, d\tau$$

implies

$$\|\phi(t) - \psi(t)\| \leq c_1 \exp\left(\|A(t)\|_{\infty,[t_0,t]}(t - t_0)\right)$$

This is true for all $c_1 \geq 0$, so set $c_1 = 0 \ldots$ □