AA/EE547: Wi 22

Module 0: Introduction to Dynamical Systems

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Purpose of these notes: The purpose of these lecture notes is to review basics of dynamical systems as models for physical phenomena. It should be a review for you.

References/Reading: Chapter 1-4 **[JH]**, Chapter 1, 5 **[C&D]**.

1 What are dynamical systems?

We are interested in studying the dynamical behavior of systems in order to analyze and control them. Examples of systems exhibiting dynamical behavior, with possible *inputs* to be controlled and *outputs* to be measured, include:

system	input	output
steam engine	amount of fuel	engine speed
aircraft	rudder and elevator angles	altitude
quadrotor	thrust & torque	position
hard drive disks	motor voltage	disk speed
rainforest ecology	yearly rainfall	population of species of interest
cooling system	flow-rate of coolant	temperature along cooling line

To study such systems we need a **model**. *What is in a model?*

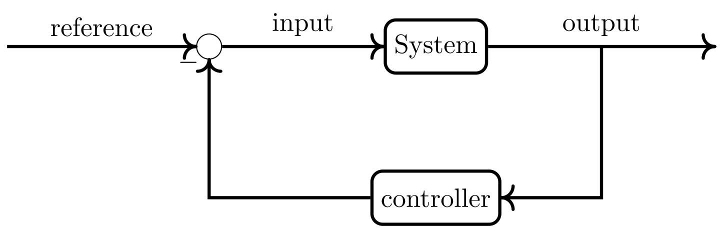
- A mathematically precise description of system behavior
- Dynamic models specify how system changes over time
- Balancing act:
  - Can't model movement of every atom of a system
  - Therefore, make simplifications
  - Need to retain essential properties (salient features) (e.g., a model that predicts negative population, or coolant temperature below absolute zero, is probably not useful)

1.1 Beyond Classical Control?

This course is about linear systems theory and a core aspect of that is controlling linear systems.

Control Theory: focuses on modeling systems and designing inputs to adjust behavior—e.g., stabilize, track a trajectory, etc.

Classical control (developed largely pre-1960s) largely adopts an input-output approach:



Key theoretical tool: Fourier/Laplace transform (i.e. Frequency domain analysis—root locus, frequency response)
Return to differential equations beginning in the '60s to address:

- numerical simulation
- many inputs/outputs
- ill-defined inputs/outputs
- non-linearities
- optimality

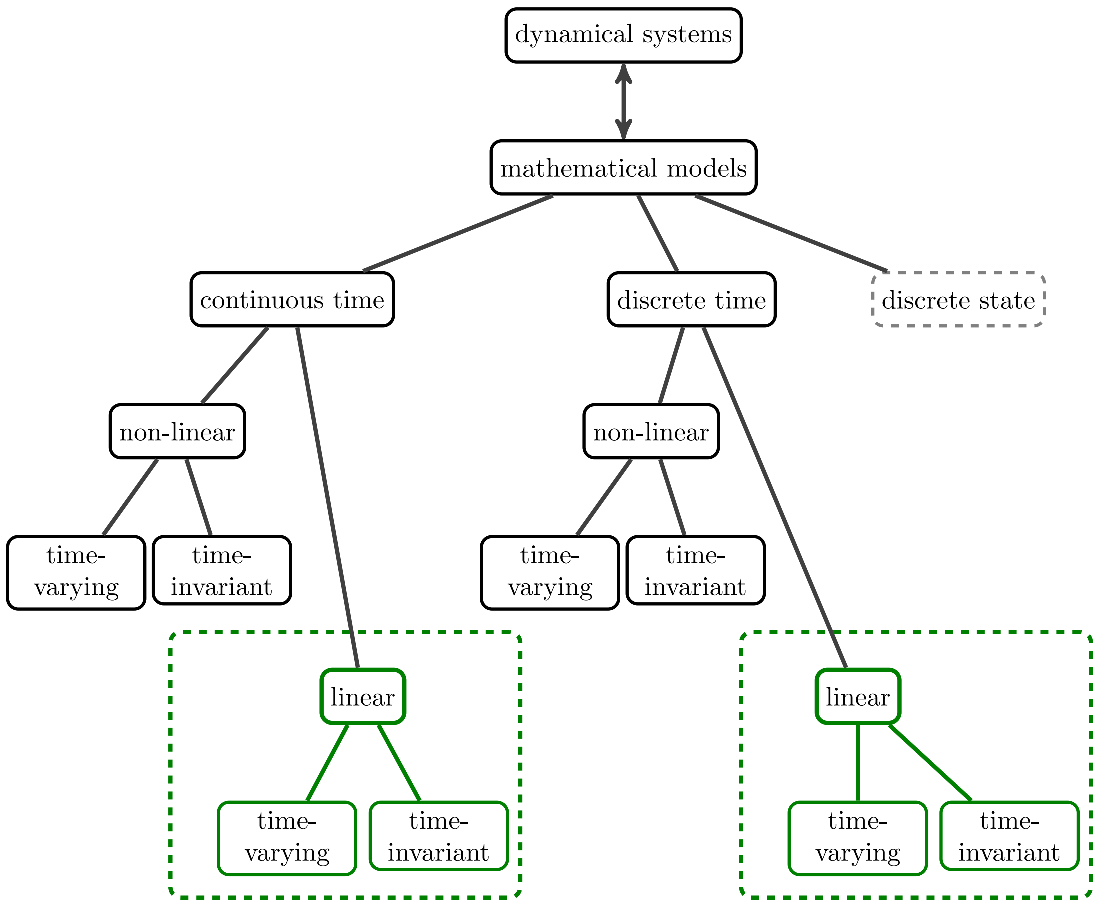
Modern control theory (∼1950's), i.e. state-space approach: Overcame some limitations of classical control enabling control of fighter jets, e.g. (related "state space" approach to ODE's is over 100 years old; control theorist just adopted it)

- System/model state is defined to capture all relevant info about past
- State often denoted by $x \in \mathbb{R}^n$, where n is state-space dimension.

What is the state for the above examples?

- engine speed/velocity
- position and yaw/pitch/roll and velocities (aircraft and quadrotor)
- disk speed/velocity
- species population, food supply, predator population, etc.
- temperature along cooling line

1.2 Dynamical Systems (non-comprehensive) Taxonomy



This course: given linear differential or difference equations, conduct rigorous analysis and design.

(This is not comprehensive as there are other types of systems combining various aspects in the diagram; however, this picture give a bit of a sense of broad categories of dynamical systems)

2 Finite Dimensional Systems as Models

How to describe a dynamical system? Some options:

- Database/look-up table containing all inputs and resulting outputs. *(What if output depends on input history? What if desired input is not in table?)*
- Function/routine in computer code
- Set of mathematical equations

As indicated in our diagram, we will study continuous-time and discrete-time finite-dimensional systems described by ordinary differential equations or difference equations.

So, in continuous time this might look like. . .

Continuous-Time: ($t \in \mathbb{R}_+ := [0, \infty)$)

$$\begin{aligned} \dot{x} &= f(t, x, u), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \\ y &= g(t, x, u), \quad y \in \mathbb{R}^p \end{aligned}$$

We have the following nomenclature:

- x is the state,
- u is the (control) input,
- y is the output (observation)

Discrete-Time: ($k \in \mathbb{N} = \{0, 1, 2, \dots\}$)

$$\begin{aligned} x[k+1] &= f(k, x, u), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \\ y[k] &= g(k, x, u), \quad y \in \mathbb{R}^p \end{aligned}$$

What about apparently more exotic systems with higher order derivatives?
For instance, consider

$$z^{(n)} = f(t, z, z^{(1)}, z^{(2)}, \dots, z^{(n-1)})$$

where $z^{(n)}$ indicates the n–th derivative of the function $z(t)$. For simplicity, assume $z \in \mathbb{R}$. Define new state variables

$$x_1 = z, \ x_2 = z^{(1)}, \ \dots, \ x_n = z^{(n-1)}$$

Then, we have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= f(t, x_1, x_2, \dots, x_n) \end{aligned}$$

Thus, without loss of generality (WLOG), we study first-order differential equations.

Example Systems: Vibrating Springs

We consider the motion of an object with mass at the end of a spring that is either vertical (as in Figure 1) or horizontal on a level surface (as in Figure 2). Hooke's Law says that if the spring is stretched (or compressed) z units from its natural length, then it exerts a force that is proportional to z—that is,

$$\text{restoring force} = -kx$$

where k is a positive constant called the spring constant. If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

$$m\ddot{x} = -kx \quad \text{or} \quad m\ddot{x} + kx = 0$$

Let $x_1 = z$ and $x_2 = \dot{z}$. Then, $\dot{x}_1 = \dot{z} = x_2$ and $\dot{x}_2 = \ddot{z} = -\frac{k}{m}z = -\frac{k}{m}x_1$. Hence,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 \end{aligned}$$

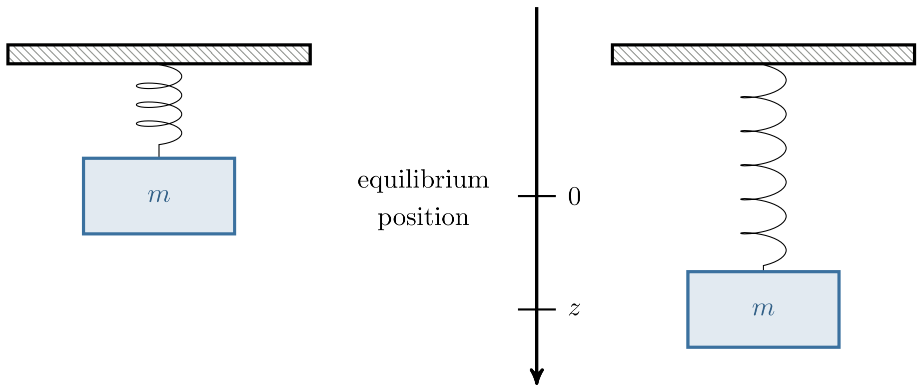


Figure 1: Vertical Pull

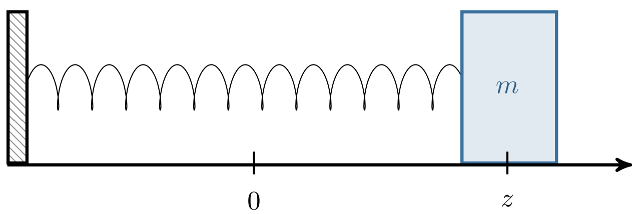


Figure 2: Horizontal Pull

In this course, we will focus on (finite-dimensional) **linear time-varying (LTV)** systems. So what do these look like notationally:

Continuous Time:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m \\ y &= C(t)x + D(t)u, \quad y \in \mathbb{R}^p \end{aligned}$$

where

- $t \in \mathbb{R}$: time
- $x(t) \in \mathbb{R}^n$: state (vector)
- $u(t) \in \mathbb{R}^m$: input or control
- $y(t) \in \mathbb{R}^p$: output
- $A(t) \in \mathbb{R}^{n \times n}$: dynamics (matrix)
- $B(t) \in \mathbb{R}^{n \times m}$: input matrix
- $C(t) \in \mathbb{R}^{p \times n}$: output or sensor matrix
- $D(t) \in \mathbb{R}^{p \times m}$: Feedthrough matrix

Equations are often written as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- A CT LDS is a first order vector differential equation
- also called state equations or m-input, n-state, p-output LDS

Discrete Time:

$$\begin{aligned} x[k+1] &= A[k]x[k] + B[k]u[k], \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m \\ y[k] &= C[k]x[k] + D[k]u[k], \quad y \in \mathbb{R}^p \end{aligned}$$

Finally, we will further specialize our results to **linear time-invariant (LTI)** systems.

Continuous Time LTI:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m \\ y &= Cx + Du, \quad y \in \mathbb{R}^p \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$ are static matrices.

Discrete Time LTI:

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k], \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m \\ y[k] &= Cx[k] + Du[k], \quad y \in \mathbb{R}^p \end{aligned}$$

Some other points:

- most linear systems encountered are time-invariant: A, B, C, D are constant as above—i.e. don't depend on t
- when there is no input u (hence, no B or D) system is called autonomous
- very often there is no feedthrough—i.e. $D = 0$
- when $u(t)$ and $y(t)$ are scalar, the system is called single-input, single-output (SISO); when input & output signal dimensions are more than one, MIMO

History Lesson:

- parts of LDS theory can be traced to 19th century
- builds on classical circuits & systems (1920s on) (transfer functions . . .) but with more emphasis on linear algebra
- first engineering application: aerospace, 1960s
- transitioned from specialized topic to ubiquitous in 1980s (just like digital signal processing, information theory, . . .)

Many dynamical systems are nonlinear, yet

- most techniques for nonlinear systems are based on linear methods; e.g., linearization to determine stability or to construct extended Kalman Filters. . .
- methods for linear systems often work unreasonably well, in practice, for nonlinear systems
- if you don't understand linear dynamical systems you certainly can't understand nonlinear dynamical systems