All hw should be uploaded to canvas as a *pdf*. Make sure that if you scan your handwritten notes that they are legible and appropriately oriented. If you use an online resource to solve any problem, please appropriately cite that source.

Problem 1. (Comparison Lemma) The following is a useful result in LQR and is used to prove that the LQR cost to go at time $t = 0$ is minimal. Prove the following statement: If $W \geq 0$ and $Q_2 \geq Q_1 \geq 0$ then P_1 and P_2 are such that $P_2 \geq P_1$ if $A - WP_2$ is asymptotically stable where P_1 and P_2 are solutions to

$$\begin{aligned} A^\top P_1 + P_1 A - P_1 W P_1 + Q_1 &= 0 \\ A^\top P_2 + P_2 A - P_2 W P_2 + Q_2 &= 0, \end{aligned}$$

Problem 2. (LQR Implementation) Consider controlling a satellite in circular orbit. The satellite is of mass m with thrust in the radial direction u_1 and in the tangential direction u_2. In cylindrical coordinates the dynamics are

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2) &= u_1 - \frac{km}{r^2} \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) &= u_2 \end{aligned}$$

a. What is the state space representation of this system?

b. Find the equilibria (i.e., where $\ddot{r} = \ddot{\theta} = 0$) when $u_1 = 0 = u_2$. Linearize about the equilibrium where $\dot{r} = 0$.

c. Given $m = 100$kg, an equilibrium radius $6.37 \times 10^3 + 300$km (first term is the radius of the earth) and $k = GM$ where $G = 6.673 \times 10^{-11}$ is the universal gravitational constant and $M = 5.98 \times 10^{24}$ is the mass of the earth, find the solution to the minimum norm control plus state LQR problem with $R = \rho I$ where $\rho = 1e6$ and $Q = I$. submit your Python notebook. Plot the state trajectories of the system and the control input overtime.

Problem 3. (Hewer's Algorithm) Consider the LQR problem

$$J(x_0, u) = \sum_{n=0}^{\infty} x_n^\top C^\top C x_n + u_n^\top R u_n$$

with

$$x_{n+1} = Ax_n + Bu_n$$

where $R > 0$ and $C^\top C > 0$. We know the optimal feedback law is

$$u^*(x) = -(B^\top P B + R)^{-1} B^\top P A x_n$$

where P is the unique positive definite solution of

$$P = A^\top P A - A^\top P B (B^\top P B + R)^{-1} B^\top P A + C^\top C$$

Let V_k, $k = 0, 1, \ldots$, be the solutions of the equation

$$V_k = (A_k)^\top V_k A_k + L_k^\top R L_k + C^\top C$$

where

$$L_k = (B^\top V_{k-1} B + R)^{-1} B^\top V_{k-1} A, \; k = 1, 2, \dots$$

and

$$A_k = A - BL_k, \; k = 0, 1, 2, \dots$$

and L_0 is chosen such that A_0 is a stability matrix. Prove the following statement:

$$K \leq V_{k+1} \leq V_k \cdots, \; k = 0, 1, \dots$$

and

$$\lim_{k \to \infty} V_k = K$$