All hw should be uploaded to canvas as a *pdf*. Make sure that if you scan your handwritten notes that they are legible and appropriately oriented. If you use an online resource to solve any problem, please appropriately cite that source.

Problem 1. (State vs. Output Feedback.) Consider a dynamical system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 7 & -4 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

and the two possible control inputs

a. $u = -[f_1\ f_2]x$

b. $u = -ky$

For each of the two control inputs above, derive a state space representation of the resulting closed loop system, and determine the characteristic equation of the resulting closed loop system matrix A_{cl}.

Problem 2. (LTV Controllability.) Given a linear time varying system $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$, show that if $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ is completely controllable on $[t_0, t_1]$, then $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ is completely controllable on any $[t_0', t_1']$, where $t_0' \leq t_0 < t_1 \leq t_1'$. Show that this is no longer true when the interval $[t_0, t_1]$ is not a subset of $[t_0', t_1']$.

Problem 3. (Echo Canceller.) This problem addresses the design of a chip that is used to cancel echo on your telephone line. The echo $y(t) \in \mathbb{R}$ is represented as the linear combinations of delayed versions of your spoken (message signal) $m(t)$ as follows:

$$y(t) = \sum_{i=1}^{N} a_i m(t - i)$$

where t is a discrete time variable, representing the sampling rate of the voice signal (around 125 micro seconds). The coefficients $a_i \in \mathbb{R}$ model the characteristics of the line. These are assumed unknown when you pick up the telephone at $t = 0$, but you have estimates of them denoted $\hat{a}_i(t)$. The aim of the echo canceller is to update the estimates using the measurement of the echo $y(t)$, and the prediction error

$$e(t) = y(t) - \sum_{i=1}^{N} \hat{a}_i m(t - i)$$

Note that it will take N seconds after you pick up the phone to get all the $m(t - i)$, thus the echo canceller is initialized with $m(-1) = m(-2) = \cdots = m(-N + 1) = 0$.

a. For this problem, set the echo canceller up as an observability problem, with the vector $a \in \mathbb{R}^N$ representing the unknown (but constant) state vector to be estimated, no input, and $y(t)$ as the output function. Remember as you construct the corresponding matrices for the observability problem, that this is a discrete time system.

b. Find $\hat{a}(1)$ so that it is the vector closest to $\hat{a}(0)$ in norm, that gives the correct value of $y(1)$.

c. Try to make this recursive, so that you can determine $\hat{a}(t+1)$ from $\hat{a}(t)$, $y(t)$.

Problem 4. (LTV Observability.) Consider the *not necessarily observable* LTV system

$$\begin{aligned} \dot{x} &= A(t)x(t) \\ y(t) &= C(t)x(t) \end{aligned}$$

with initial condition x_0 at time 0.

a. Suppose we observe output $y(t)$ over the interval $[0, t_1]$. Under what conditions can we determine the initial state x_0? Justify your answer and provide an expression for x_0.

b. Consider the measure of the energy of the output defined by $\|y\|^2$. Provide an expression for the energy in terms of x_0.

c. Consider all initial conditions x_0 such that $\|x_0\| = 1$. Is it possible for $\|y\|^2$ to be zero? Justify your answer.

Problem 5. (How does state feedback impact observability?) Consider the system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

and assume that you would like to design a feedback controller of the form $u = -Fx + r$, where r is a reference input signal.

a. Show that the system is observable.

b. Show that there exists a state feedback gain matrix $F = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$ such that the closed loop system resulting from setting $u = -Fx + r$ is not observable.

c. Now, compute a matrix F of the form $F = \begin{bmatrix} 1 & f_2 \end{bmatrix}$ such that the closed loop system (as in part 2) is not observable.

d. The transfer function of a system is

$$H(s) = C(s - IA)^{-1}B$$

By comparing the open loop transfer function with the transfer function of the closed loop system of part c., state what this unobservability is due to?

Problem 6. (Connections between Lyapunov and Observability.) Suppose that $A, Q \in \mathbb{F}^{n\times n}$ are given. Assume that A is stable and $Q = Q^* \geq 0$. Prove the following claim:
The pair (A, Q) is observable if and only if

$$X = \int_0^\infty e^{A^*\tau} Q e^{A\tau}\, d\tau > 0$$