All hw should be uploaded to canvas as a *pdf*. Make sure that if you scan your handwritten notes that they are legible and appropriately oriented. If you use an online resource to solve any problem, please appropriately cite that source.

Problem 1. (Comparison Lemma) The following is a useful result in LQR and is used to prove that the LQR cost to go at time $t = 0$ is minimal. Prove the following statement: If $W \geq 0$ and $Q_2 \geq Q_1 \geq 0$ then P_1 and P_2 are such that $P_2 \geq P_1$ if $A - WP_2$ is asymptotically stable where P_1 and P_2 are solutions to

$$\begin{aligned} A^\top P_1 + P_1 A - P_1 W P_1 + Q_1 &= 0 \\ A^\top P_2 + P_2 A - P_2 W P_2 + Q_2 &= 0, \end{aligned}$$

Solution. This is called the comparison lemma. Such comparisons are often made in stability analysis for dynamical systems amongst other analyses. First, observe that

$$\begin{aligned} A^\top P_1 + P_1 A - P_1 W P_1 + Q_1 = (A - WP_2)^\top P_1 + P_1(A - WP_2) + P_2 W P_2 \\ + Q_1 - (P_1 - P_2) S (P_1 - P_2) \end{aligned}$$

and

$$A^\top P_2 + P_2 A - P_2 W P_2 + Q_2 = (A - WP_2)^\top P_2 + P_2(A - WP_2) + P_2 W P_2 + Q_2$$

Subtracting the above equations we get a Lyapunov equation:

$$(A - WP_2)^\top \Delta P + \Delta P(A - WP_2) + \Delta Q$$

where

$$\Delta P = X_2 - X_1$$

and

$$\Delta Q = Q_2 = Q_1 + (P_1 - P_2) W (P_1 - P_2)$$

Since $\Delta Q \geq 0$, if $A - WP_2$ is Hurwitz (stable) we conclude that $\Delta P \geq 0$ so that $P_2 \geq P_1$.

Problem 2. (LQR Implementation) Consider controlling a satellite in circular orbit. The satellite is of mass m with thrust in the radial direction u_1 and in the tangential direction u_2. In cylindrical coordinates the dynamics are

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2) &= u_1 - \frac{km}{r^2} \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) &= u_2 \end{aligned}$$

a. What is the state space representation of this system?

b. Find the equilibria (i.e., where $\ddot{r} = \ddot{\theta} = 0$) when $u_1 = 0 = u_2$. Linearize about the equilibrium where $\dot{r} = 0$.

c. Given $m = 100$kg, an equilibrium radius $6.37 \times 10^3 + 300$km (first term is the radius of the earth) and $k = GM$ where $G = 6.673 \times 10^{-11}$ is the universal gravitational constant and $M = 5.98 \times 10^{24}$ is the mass of the earth, find the solution to the minimum norm control plus state LQR problem with $R = \rho I$ where $\rho = 1e6$ and $Q = I$. submit your Python notebook. Plot the state trajectories of the system and the control input overtime.

Solution.

a. Let

$$x = \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix}$$

so that

$$\dot{x} = \begin{bmatrix} x_3 \\ x_4 \\ x_1 x_4^2 - \frac{k}{x_1^2} \\ -2\frac{x_3 x_4}{x_1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{m x_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

b. With $\ddot{r} = \ddot{\theta} = 0$ and $u_i = 0$, to find an equilibrium we need to solve

$$x_1 x_4^2 - \frac{k}{x_1} = 0, \; -2x_3 \frac{x_4}{x_1} = 0$$

Doing so gives us that $x_3 = \dot{r}$ and or $x_4 = \dot{\theta} = 0$. Let $x_3 = 0$ so that $x_1 = \tilde{r}$ is some constant and

$$x_4 = \dot{\theta} = \sqrt{\frac{k}{x_1^3}} = \tilde{\omega} \implies k = \tilde{r}^3 \tilde{\omega}^2$$

and $x_2 = \theta = \tilde{\omega} t$. Linearizing we have that

$$\tilde{x}(t) = \begin{bmatrix} \tilde{r} \\ \tilde{\omega} t \\ 0 \\ \tilde{\omega} \end{bmatrix},$$

so that

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2k}{\tilde{r}^3} + \tilde{\omega}^2 & 0 & 0 & 2\tilde{r}\tilde{\omega} \\ 0 & 0 & -2\frac{\tilde{\omega}}{\tilde{r}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\tilde{\omega}^2 & 0 & 0 & 2\tilde{r}\tilde{\omega} \\ 0 & 0 & -2\frac{\tilde{\omega}}{\tilde{r}} & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{m\tilde{r}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

If we looking at the satellite (from the earth) we can say that we can observe r and $\dot{\theta}$ (distance and angular speed) so that

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tilde{x}$$

c. see Python notebook

Problem 3. (Hewer's Algorithm) Consider the LQR problem

$$J(x_0, u) = \sum_{n=0}^{\infty} x_n^\top C^\top C x_n + u_n^\top R u_n$$

with

$$x_{n+1} = A x_n + B u_n$$

where $R > 0$ and $C^\top C > 0$. We know the optimal feedback law is

$$u^*(x) = -(B^\top PB + R)^{-1}B^\top PAx_n$$

where P is the unique positive definite solution of

$$P = A^\top PA - A^\top PB(B^\top PB + R)^{-1}B^\top PA + C^\top C$$

Let V_k, $k = 0, 1, \ldots$, be the solutions of the equation

$$V_k = (A_k)^\top V_k A_k + L_k^\top RL_k + C^\top C$$

where

$$L_k = (B^\top V_{k-1}B + R)^{-1}B^\top V_{k-1}A, \; k = 1, 2, \ldots$$

and

$$A_k = A - BL_k, \; k = 0, 1, 2, \ldots$$

and L_0 is chosen such that A_0 is a stability matrix. Prove the following statement:

$$K \le V_{k+1} \le V_k \cdots, \; k = 0, 1, \ldots$$

and

$$\lim_{k\to\infty} V_k = K$$

Solution. Since A_0 is a stability matrix, we know that

$$V_0 = \sum_{\ell=0}^{\infty}(A_0^\top)^\ell (L_0^\top RL_0 + C^\top C)A_0^\ell$$

is unique and positive definite and solves the recursion

$$V_k = (A_k)^\top V_k A_k + L_k^\top RL_k + C^\top C \qquad (*)$$

Let L_1 be defined by the above expression for L_k and consider the identity

$$A_0^\top V_0 A_0 + L_0^\top RL_0 = A_1^\top V_0 A_1 + L_1^\top RL_1 + (L_1 - L_0)^\top (B^\top V_0 B + R)(L_1 - L_0) \qquad (**)$$

(i.e. like the comparison lemma above). Then by this expression V_0 also satisfies

$$V_0 = A_1^\top V_0 A_1 + M$$

where

$$M = C^\top C + L_1^\top RL_1 + (L_1 - L_0)^\top (B^\top V_0 B + R)(L_1 - L_0) > 0$$

Since this implies that A_1 is a stability matrix, the unique positive definite solution V_1 of $(*)$ exists. Using $(**)$ with V_0 and V_1 given by $(*)$ we have

$$V_1 - V_0 = \sum_{n=0}^{\infty}(A_0^n)^\top (L_0 - L_1)^\top (B^\top V_0 B + R)(L_0 - L_1)A_0^n \ge 0$$

and hence $V_1 \le V_0$. (we are basically applying the comparison lemma). Now, let $L^* = (B^\top KB + R)^{-1}B^\top KA$ which is well defined by the choice of R. By employing an identity similar to $(**)$ we get

$$V_1 - K = \sum_{n=0}^{\infty}(A_0^n)^\top (L_0 - L^*)^\top (B^\top KB + R)(L_0 - L^*)A_0^n \ge 0$$

so that

$$V_1 \geq K.$$

Hence V_1 is also bounded below and therefore has finite norm. Thus A_1 has eigenvalues with negative real parts, and so V_1 satisfies $(*)$ with $k = 1$. Repeating the above argument for $k = 2, 3, \ldots$ yields the desired result. Now

$$\lim_{k\to\infty} V_k = V_\infty$$

exists (by the monotonic convergence theorem for positive operators) so that by taking the limit of $(*)$ as $k \to \infty$ we get the typical discrete time steady state Riccati equation which agrees with our construction of L^* in terms of K.