All hw should be uploaded to canvas as a *pdf*. Make sure that if you scan your handwritten notes that they are legible and appropriately oriented. If you use an online resource to solve any problem, please appropriately cite that source.

Problem 1. (State vs. Output Feedback.) Consider a dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 7 & -4 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

and the two possible control inputs

a. $u = -[f_1\ f_2]x$

b. $u = -ky$

For each of the two control inputs above, derive a state space representation of the resulting closed loop system, and determine the characteristic equation of the resulting closed loop system matrix A_{cl}.

Solution.

a. The closed loop A is given by

$$A_{cl} = A - B\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} -f_1 & 1 - f_2 \\ 7 - 2f_1 & -4 - 2f_2 \end{bmatrix}$$

The characteristic equation is

$$\chi_{A_{cl}}(s) = s^2 + (4 + 2f_2 + f_1)s + 6f_1 + 7f_2 - 7$$

The state space representation is then

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -f_1 & 1 - f_2 \\ 7 - 2f_1 & -4 - 2f_2 \end{bmatrix} x \\ y &= \begin{bmatrix} 1 & 3 \end{bmatrix} x \end{aligned}$$

b. The closed loop A is given by

$$A_{cl} = A - kBC = \begin{bmatrix} -k & 1 - 3k \\ 7 - 2k & -4 - 6k \end{bmatrix}$$

and the characteristic equation is

$$\chi_{A_{cl}}(s) = s^2 + (7k + 4)s + 27k - 7$$

The state space representation is then

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -k & 1 - 3k \\ 7 - 2k & -4 - 6k \end{bmatrix} x \\ y &= \begin{bmatrix} 1 & 3 \end{bmatrix} x \end{aligned}$$

Problem 2. (LTV Controllability.) Given a linear time varying system $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$, show that if $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ is completely controllable on $[t_0, t_1]$, then $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ is completely controllable on any $[t_0', t_1']$, where $t_0' \leq t_0 < t_1 \leq t_1'$. Show that this is no longer true when the interval $[t_0, t_1]$ is not a subset of $[t_0', t_1']$.

Solution. Recall that $\Phi(t, \tau)$ is always invertible. To steer the system from x to y on $[t_0', t_1']$ where $[t_0, t_1] \subset [t_0', t_1']$, choose an input that steers the system from $\Phi(t_0, t_0')x$ to $\Phi(t_1', t_1)^{-1}y$ on $[t_0, t_1]$. As a counter example for the case when $[t_0, t_1]$ is not a subset of $[t_0', t_1']$, consider a controllable system for which $B(\sigma) \equiv 0$ for $\sigma \in [t_0', t_1']$.

Problem 3. (Echo Canceller.) This problem addresses the design of a chip that is used to cancel echo on your telephone line. The echo $y(t) \in \mathbb{R}$ is represented as the linear combinations of delayed versions of your spoken (message signal) $m(t)$ as follows:

$$y(t) = \sum_{i=1}^{N} a_i m(t-i)$$

where t is a discrete time variable, representing the sampling rate of the voice signal (around 125 micro seconds). The coefficients $a_i \in \mathbb{R}$ model the characteristics of the line. These are assumed unknown when you pick up the telephone at $t = 0$, but you have estimates of them denoted $\hat{a}_i(t)$. The aim of the echo canceller is to update the estimates using the measurement of the echo $y(t)$, and the prediction error

$$e(t) = y(t) - \sum_{i=1}^{N} \hat{a}_i m(t-i)$$

Note that it will take N seconds after you pick up the phone to get all the $m(t-i)$, thus the echo canceller is initialized with $m(-1) = m(-2) = \cdots = m(-N+1) = 0$.

a. For this problem, set the echo canceller up as an observability problem, with the vector $a \in \mathbb{R}^N$ representing the unknown (but constant) state vector to be estimated, no input, and $y(t)$ as the output function. Remember as you construct the corresponding matrices for the observability problem, that this is a discrete time system.

b. Find $\hat{a}(1)$ so that it is the vector closest to $\hat{a}(0)$ in norm, that gives the correct value of $y(1)$.

c. Try to make this recursive, so that you can determine $\hat{a}(t+1)$ from $\hat{a}(t)$, $y(t)$.

Solution. Let t be a discrete time variable, $y[t] = \sum_{i=1}^{N} a_i m[t-i]$ and $e[t] := y[t] - \sum_{i=1}^{N} \hat{a}_i m[t-i]$. Let the echo canceller be initialized with $m[-1] = m[2] = \cdots = m[-N+1] = 0$.

a. The state space model is: $a[t+1] = I \cdot a[t]$ and $y[t] = C[t]a[t] = [m[t-1] \ \cdots \ m[t-N]][a_1[t] \ \cdots \ a_N[t]]^T$. Now, the observability map is defined by $\mathcal{L}_o = C(\cdot)\Phi(\cdot, t) = C[\cdot]$ since $\Phi(\cdot, t) = I$. Using $\mathcal{L}_o^*$, given $y[t]$ we can uniquely recover the state:

$$\mathcal{L}_o^* \begin{bmatrix} y[t] \\ \vdots \\ y[t+N] \end{bmatrix} = \mathcal{L}_o^* \begin{bmatrix} C[t] \\ \vdots \\ C[t+N] \end{bmatrix} x[t]$$

b. To find $\hat{a}[1]$ that is the vector closest to $\hat{a}[0]$ in norm that gives the correct value of $y[1]$, we need $e[1] = 0$ and $\|\hat{a}[1] - \hat{a}[0]\|$ needs to be minimized. If we consider $e[1] = 0$, we get

$$0 = e[1] = y[1] - \sum_{i=1}^{N} \hat{a}_i m[1-i] = y[1] - m[0]\hat{a}_1[1]$$

since $m[-1] = m[2] = \cdots = m[-N+1] = 0$. Thus, $\hat{a}_1[1] = \frac{y[1]}{m[0]}$. Now, assuming a given $\hat{a}[0]$, $\hat{a}_1[1] - \hat{a}_1[0] = \frac{y[1]}{m[0]} - \hat{a}_1[0]$. So, in order to minimize the norm $\|\hat{a}[1] - \hat{a}[0]\|$, we can set $\hat{a}_i[1]$ equal to $\hat{a}_i[0]$ for all $i \in \{2, \ldots, N\}$ and $\hat{a}_1[1] - \hat{a}_1[0] = \frac{y[1]}{m[0]} - \hat{a}_1[0]$.

c. Now, to set this up as a recursive problem, at each time step we can update the corresponding entry of $\hat{a}$. As seen in part b, the initialization allows us to do this. So, consider $t = 2$. Then we want $e[1] = e[2] = 0$ and $\|\hat{a}[2] - \hat{a}[1]\|$ to be minimized. So, by part b, the condition $e[1] = 0 \Rightarrow \hat{a}_1[1] = \frac{y[1]}{m[0]}$. And, $0 = e[2] = y[2] - \hat{a}_1[2]m[1] - \hat{a}_2[2]m[0] \Rightarrow \hat{a}_2[2] = \frac{y[2]-\hat{a}_1[2]m[1]}{m[0]}$. In order to minimize the norm, we have the following conditions: $\hat{a}_i[2] = \hat{a}_i[1]$ if $i \neq 2$ and

$$\hat{a}_2[2] = \frac{y[2] - \hat{a}_1[1]m[1]}{m[0]}$$

(note that $\hat{a}_1[2] = \hat{a}_1[2]$). This allows us to set up a recursive formula for estimating $\hat{a}[t+1]$ from $\hat{a}[t]$ and $y[t]$: $\hat{a}_i[t+1] = \hat{a}_i[t]$ if $i \neq (t+1)$ and

$$\hat{a}_i[t+1] = \frac{y[t+1] - \sum_{j=1}^{t} m[(t+1)-j]\hat{a}_j[t]}{m[0]}$$

if $i = (t+1)$

Problem 4. (LTV Observability.) Consider the *not necessarily observable* LTV system

$$\begin{aligned} \dot{x} &= A(t)x(t) \\ y(t) &= C(t)x(t) \end{aligned}$$

with initial condition x_0 at time 0.

a. Suppose we observe output $y(t)$ over the interval $[0, t_1]$. Under what conditions can we determine the initial state x_0? Justify your answer and provide an expression for x_0.

b. Consider the measure of the energy of the output defined by $\|y\|^2$. Provide an expression for the energy in terms of x_0.

c. Consider all initial conditions x_0 such that $\|x_0\| = 1$. Is it possible for $\|y\|^2$ to be zero? Justify your answer.

Solution.

a. We may solve for the state $x(t)$ to get

$$x(t) = \Phi(t, t_0)x_0$$

The output is simply $y(t) = C(t)\Phi(t, t_0)x_0$. In order to determine the initial state x_0, we need only to ensure that $y(t) \in \mathcal{R}(C(t)\Phi(t, t_0))$ for all $t \in [0, t_1]$. Let us define the operator

$\mathcal{L}_0 : x_0 \mapsto C(t)\Phi(t,t_0)x_0$. Then x_0 may be determined iff $y(t) \in \mathcal{R}(\mathcal{L}_0)$ for all $t \in [0, t_1]$. To derive an expression for x_0, we start with $\mathcal{L}_0 x_0 = y$ and

$$\mathcal{L}_0 x_0 = y \implies \mathcal{L}_0^* \mathcal{L}_0 x_0 = \mathcal{L}_0^* y \implies x_0 = (\mathcal{L}_0^* \mathcal{L}_0)^{-1} \mathcal{L}_0^* y$$

And, x_0 is unique if $\mathcal{N}(\mathcal{L}_0^* \mathcal{L}_0) = \{0\}$.

b. The energy of the output is given by

$$\langle y, y \rangle = x_0^* W_0(t_0, t_1) x_0$$

where

$$W_0(t_0, t_1) = \int_{t_0}^{t_1} \Phi^*(\tau, t_0) C^*(\tau) C(\tau) \Phi(\tau, t_0) \, d\tau$$

c. First, the observability gramian $W_0(t_0, t_1)$ is positive semi-definite. If (A, C) were observable, then this gramian would be positive definite and the energy would never be zero for $\|x_0\| = 1$. However, since (A, C) is not necessarily obserable, it is possible for x_0 to lie in the nullspace of W_0 leading to zero energy. Mathematically,

$$\|y\|^2 = \langle y, y \rangle = x_0^* W_0(t_0, t_1) x_0 \le \|x_0^*\| \|W_0(t_0, t_1)\| \|x_0\| = \|W_0(t_0, t_1)\|$$

When is the right-hand side zero? Since $W_0(t_0, t_1) \ge 0$, the only way for $W_0(t_0, t_1) = 0$ is for it to have a non-trivial null space which occurs for x_0 unobservable since $\mathcal{N}(W_0(t_0, t_1)) = \mathcal{N}(\mathcal{L}_0)$ which is possible if the system is not observable.

Problem 5. (How does state feedback impact observability?) Consider the system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

and assume that you would like to design a feedback controller of the form $u = -Fx + r$, where r is a reference input signal.

a. Show that the system is observable.

b. Show that there exists a state feedback gain matrix $F = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$ such that the closed loop system resulting from setting $u = -Fx + r$ is not observable.

c. Now, compute a matrix F of the form $F = \begin{bmatrix} 1 & f_2 \end{bmatrix}$ such that the closed loop system (as in part 2) is not observable.

d. The transfer function of a system is

$$H(s) = C(s - IA)^{-1} B$$

By comparing the open loop transfer function with the transfer function of the closed loop system of part c., state what this unobservability is due to?

Solution.

a. Since the system is LTI and $x \in \mathbb{R}^2$ so that $n = 2$, we can consider the following observability matrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Now, $\operatorname{rank} \mathcal{O} = 2 = n$ so that the system is observable.

b. If we let the state feedback be $u = -Fx + r$ with $F = [f_1 \;\; f_2]$, then we have $\dot{x} = (A - bF)x + br$ where

$$A - bF = \begin{bmatrix} -2 - f_1 & 1 - f_2 \\ 1 & 0 \end{bmatrix}$$

Now, from lecture we have the following theorem: The LTI system is (completely observable) CO on some $[0, \Delta] \Leftrightarrow \operatorname{rank} \mathcal{O} = n$. So, we must consider the following $\mathcal{O}$:

$$\mathcal{O} = \begin{bmatrix} C \\ C(A - bF) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -f_1 & 1 - f_2 \end{bmatrix}$$

Considering the converse of the theorem, we have that the feedback system is not observable $\Leftrightarrow$ $1 - f_2 + 2f_1 = 0$ (i.e. if the determinant is 0 then O is not of full rank).

c. Now, let $F = [-1 \;\; -f_2]$. The closed-loop system will not be observable if $1 = f_2$ using the above condition.

d. If we consider $F = [-1 \;\; -1]$, the closed loop dynamics are

$$A_{cl} = A - bF = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$$

so that

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r \\ y &= \begin{bmatrix} 12 \end{bmatrix} x \end{aligned}$$

Then the closed loop transfer function is

$$H(s) = C(sI - A_{cl})^{-1}B = \frac{s+2}{s^2 + s - 2} = \frac{s+2}{(s-1)(s+2)} = \frac{1}{s-1}$$

and the open-loop transfer function is

$$H(s) = \frac{s+2}{s^2 + 2s - 1}$$

The pole-zero cancellation in the closed loop system makes it so that we cannot observe the state completely.

Problem 6. (Connections between Lyapunov and Observability.) Suppose that $A, Q \in \mathbb{F}^{n \times n}$ are given. Assume that A is stable and $Q = Q^* \geq 0$. Prove the following claim:
The pair (A, Q) is observable if and only if

$$X = \int_0^\infty e^{A^* \tau} Q e^{A\tau} \, d\tau > 0$$

Solution.($\Longrightarrow$): Suppose that X is not positive definite. We know just by the construction of X that it is at least positive semi-definite. So if it is not positive definite, then there exists some $x_0 \neq 0$ such that $x_0^* X x_0 = 0$. Using the integral form for X, since A is stable by assumption, and the fact that $Q \geq 0$, we have that

$$\int_0^\infty x_0 e^{A^T\tau} Q^{T/2} Q^{1/2} e^{A\tau} x_0 \; d\tau = \int_0^\infty \left\| Q^{1/2} e^{A\tau} x_0 \right\|^2 \; d\tau = 0$$

The integrand is non-negative and continuous, hence it must be zero for all $\tau \geq 0$. Thus, $Qe^{At}x_0 = 0$ for all $t \geq 0$. Then using the infinite series expansion for e^{At} and Cayley-Hamilton, we have that

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{n-1} \alpha_k(t) A^k$$

so that

$$Qe^{At}x_0 = \sum_{k=0}^{n-1} \alpha_k(t) Q A^k x_0 = 0, \; \forall t \geq 0$$

which in turn says that $QA^k x_0 = 0$ for each $k \in \{0, \ldots, n-1\}$. Hence, (A, Q) is not observable.

($\Longleftarrow$): Again suppose not. Then there is an $x_0 \neq 0$ such that

$$Qe^{At}x_0 = 0, \; \forall t \geq 0$$

(by the same argument used in the conclusion of the preceding argument). Consequently, $x_0^* e^{A^*t} Q e^{At} x_0 = 0$ for all $t \geq 0$. Integrating gives us

$$0 = \int_0^\infty (x_0^* e^{A^*t} Q e^{At} x_0) \; dt = x_0^* \left(\int_0^\infty e^{A^*t} Q e^{At} \; dt \right) x_0 = x_0^* X x_0$$

Since $x_0 \neq 0$, X is not positive definite.