All hw should be uploaded to canvas as a *pdf*. Make sure that if you scan your handwritten notes that they are legible and appropriately oriented. If you use an online resource to solve any problem, please appropriately cite that source.

Problem 1.(Numerical Integration of Conservative Systems.) A conservative physical system is modeled by $\dot{x} = Ax$, $A \in \mathbb{C}^{n\times n}$ and it is normalized so that along any trajectory, the map $t \mapsto \|x(t)\|_2$ is constant where $\|x(t)\|_2^2$ is the 'energy'—i.e., the energy is conserved.

a. Using the fact that the energy is constant, show that A is skew-symmetric and that skew-symmetric matrices have purely imaginary eigenvalues.

b. Let A be diagonalizable for simplicity. In order to integrate numerically, a student considers three methods: with step size h satisfying

$$0 < h \ll \rho(A) = \max\{|\lambda| \ : \ \lambda \in \sigma(A)\}$$

the methods are

(a) forward Euler:
$$\xi_{k+1} = (I + hA)\xi_k,\ \xi_0 = x(0)$$

(b) backward Euler:
$$\xi_{k+1} = (I - hA)^{-1}\xi_k,\ \xi_0 = x(0)$$

(c) forward/backward:
$$\xi_{k+1} = \left(I + A\frac{h}{2}\right)\left(I - A\frac{h}{2}\right)^{-1}\xi_k,\ \xi_0 = x(0)$$

Suppose we want to choose a step size $h < 2\min_i |\lambda_i(A)|$. Select a method that is the most appropriate for this problem. That is, which discretization method produces solutions that are consistent with the continuous time dynamical system behavior. Justify your choice. Towards this end, use part a. and consider the map $\|\xi_k\|$ for each method, and characterize when the map is stable (i.e., conditions under which $\|\xi_k\|$ respects the energy conservation conditions as $k \to \infty$).

c. Consider the system $\dot{x} = Ax$ where

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

(a) Use a (Python) Jupyter notebook to implement each of the methods. Choose a non-trivial (not all zero) initial condition and implement each of the schemes above and show why your choice in part b. is the right choice by plotting the iterates ξ_k for each state and the norm of the iterates ξ_k.

(b) Now, for a non-trivial initial condition, compute the exact solution $x(t) = e^{At}x_0$ and plot the exact solution and the trajectories from the three numerical methods. In addition, plot the norm of the difference between the exact solution and the numerical solution for each of methods. Vary your choice of h and show how this changes this norm for each method (i.e. assess the numerical error)?

You will submit your Jupyter notebook along with the pdf.

d. You don't need to submit this part. You may also consider playing around with the initial condition to help build more intuition. Another thing you can try to do is add very small noise ε to the diagonal entries of A. How does this impact the solution?

Solution.

a. We claim that if the solution to $\dot{x} = Ax$ has constant norm, then the eigenvalues of A are purely imaginary. Indeed, suppose that $x(t)$ is the solution to $\dot{x} = Ax$ and it has constant norm. First, we show that A is skew–symmetric (recall that we showed this in Discussion 8). The system has constant norm if and only if

$$\begin{aligned} 0 &= \frac{d}{dt}\|x(t)\|^2 \\ &= 2x(t)^*\dot{x}(t) \\ &= 2x(t)^*Ax(t) \\ &= x(t)^*(A + A^*)x(t) \end{aligned}$$

for all $x(t)$. This occurs if and only if $A + A^* = 0$ which is equivalent to $A^* = -A$, i.e. A is a skew–Hermitian matrix. Note that you can show this in many other ways. I encourage you to think about other possible ways to show this. It might be helpful to prepare you for the midterm.

Now, we show that skew–Hermitian matrices have imaginary eigenvalues. Moreover, the eigenvalues are purely imaginary (or exactly zero). Indeed, for every $x, y \in \mathbb{C}^n$, we have

$$\langle y, Ax\rangle = \langle A^*y, x\rangle = -\langle Ay, x\rangle.$$

Hence, letting $y = x$, we have

$$\langle x, Ax\rangle = \langle x, \lambda x\rangle = \lambda\|x\|^2$$

and

$$-\langle Ax, x\rangle = -\langle \lambda x, x\rangle = -\bar{\lambda}\|x\|^2.$$

So, since $\lambda\|x\|^2 = -\bar{\lambda}\|x\|^2$ for all x, $\lambda = -\bar{\lambda}$. Thus, all the eigenvalues are on imaginary axis.

b. Each method is a discrete time method of the form $\xi_{k+1} = M\xi_k$. If we can diagonalize M as $M = UDU^\top$ where $D = \text{diag}(\mu_1, \ldots, \mu_n)$, then

$$\|\xi_k\|^2 = \xi_0^T U \,\text{diag}(|\mu_1|^{2k}, \ldots, |\mu_n|^{2k})U^T\xi_0.$$

So the behavior of the squared norm of the solution depends on whether the eigenvalues have magnitude less than one, equal to one, or greater than one. Since A is skew–symmetric, we have that A commutes with its transpose, i.e. it is normal. Indeed,

$$AA^* = A(-A) = -A^2 = -AA = A^*A.$$

Therefore, we can write $A = U\Lambda U^T$ where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of A. Now, let us examine the three proposed numerical methods. For forward Euler, we have

$$M = I + hA = U(I + h\Lambda)U^\top$$

Hence, the eigenvalues of M in this case are $\mu_i = 1+h\lambda_i$. We know that λ_i is purely imaginary, so $|\mu_i|^2 > 1$. In this case, $\|\xi_k\|^2 \to \infty$ as $k \to \infty$. For backward Euler,

$$M = (I - hA)^{-1} = (U(I - h\Lambda)U^\top)^{-1} = U(I - h\Lambda)^{-1}U^\top$$

so in this case

$$\mu_i = \frac{1}{1 - h\lambda_i} \implies |\mu_i| = \frac{1}{|1 - h\lambda_i|} < 1$$

so that $\|\xi_k\|^2 \to 0$ as $k \to \infty$. For Forward-backward Euler,

$$M = \left(I - \frac{h}{2}A\right)^{-1}\left(I + \frac{h}{2}A\right)$$

so that combining the decompositions above we get

$$M = U\left(I - \frac{h}{2}\Lambda\right)^{-1}\left(I + \frac{h}{2}\Lambda\right)U^\top$$

Thus, we have

$$|\mu_i| = \frac{|1 + \frac{h}{2}\lambda_i|}{|1 - \frac{h}{2}\lambda_i|} = 1$$

since λ_i is purely imaginary. Hence the iterates generated by this method will have the same norm as ξ_0. This reflects the true solution to $\dot{x} = Ax$ and is the best method to use here.

c. See hw1s.ipynb

Problem 2. Eigenvalues and Numerical Solutions Consider the following ODE

$$\dot{x} = -20x \quad (1)$$
$$x(0) = 1 \quad (2)$$

1. For which values of h is the backward Euler scheme unstable.
2. Simulate the ODE with backward Euler scheme using a value of h for which the backward Euler scheme is stable. Include code and plots.

solution

1. All h such that
$$\left|\frac{1}{1 + h20}\right| < 1 \implies 0 < h < \infty$$
2. see hw6s.ipynb

Problem 3.(Asymptotic Stability for LTV Systems.) If $A(t) = A^\top(t) \in \mathbb{R}^{n\times n}$ and the largest eigenvalue of $A(t)$ satisfies $\lambda_{\max}(A(t)) \leq -\varepsilon$ for all t and some $\varepsilon > 0$, show that the state transition matrix of $A(t)$ is asymptotically stable.

Solution. Let $A(t) = A^T(t) \in \mathbb{R}^{n\times n}$ and $\lambda_{\max}(A(t)) \leq -\varepsilon < 0$ for all $t \in \mathbb{R}$. Consider $V(x) = x^\top x$. Then,

$$\dot{V}(x) = \dot{x}^\top x + x^\top \dot{x} = x^\top A^\top(t)x + x^\top A(t)x = 2x^\top A(t)x \leq -2\varepsilon x^\top x = -2\varepsilon V(x)$$

Thus,

$$V(x(t)) \le V(x(t_0))e^{-2\varepsilon t} \implies \|x(t)\|^2 \le \|x(t_0)\|^2 e^{-2\varepsilon t} \implies \|x(t)\| \le \|x(t_0)\| e^{-\varepsilon t}$$

Thus, the system is asymptotically stable since exponential stability implies asymptotic stability.

Problem 4. (Extended Lyapunov Analysis.) Suppose that there exist positive definite matrices $P, Q \in \mathbb{R}^{n\times n}$ and some $\lambda > 0$ such that

$$A^\top P + PA - 2\lambda P = -Q$$

What can you say about the eigenvalues of A? In particular, analytically characterize the eigenvalues of A.

Solution. For any eigenvalue α of A, we can choose its corresponding eigenvector x. Then, we obtain

$$x^* A^\top Px + x^* PAx - 2\lambda x^* Px = -x^* Qx$$

Hence,

$$2(\mathrm{Re}(\alpha) - \lambda)x^* Px = -x^* Qx < 0$$

Note that P is positive definite so that $\mathrm{Re}(\alpha) < \lambda$.

Problem 5.(DT Lyapunov Equation.) Let $\sigma(A) \subset D(0,1)$. Show that for all $Q = Q^* > 0$

$$P = A^* PA + Q$$

has a unique solution $P = P^* > 0$.

Solution. Suppose that A is stable and construct

$$P = \sum_{t=0}^{\infty} (A^*)^t Q A^t$$

The matrix P is symmetric and positive definite since Q is. We need to show that this P solves the DT Lyapunov equation. Indeed,

$$A^* \sum_{t=0}^{\infty} \left((A^*)^t Q A^t\right) A - \sum_{t=0}^{\infty} (A^*)^t Q A^t = \sum_{t=0}^{\infty} \left((A^*)^{t+1} Q A^{t+1} - (A^*)^t Q A^t\right) = -Q$$

We now need to show that P is unique. Suppose not. Then there exists some other $\tilde{P} \neq P$ solving the DT Lyapunov equation. Hence,

$$A^* PA - P = A^* \tilde{P} A - \tilde{P} \iff A^*(P - \tilde{P})A - (P - \tilde{P}) = 0$$

from which, defining $R(x) = x^*(P - \tilde{P})x$ it follows that

$$R(f(x)) - R(x) = 0,\ \forall x \in \mathbb{R}^n \iff R(x(0)) = R(x(t)),\ \forall t \ge 0$$

Now it holds that

$$\lim_{t\to\infty} R(x(t)) = \lim_{t\to\infty} x(0)^*(A^*)^t (P - \tilde{P}) A^t x(0) = 0,\ \forall x \in \mathbb{R}^n$$

since A is stable and $R(x(0)) = R(x(t))$, we have that

$$x^*(P - \tilde{P})x = 0,\ \forall x \in \mathbb{R}^n \iff P - \tilde{P} = 0$$

so that the solution is unique.