All hw should be uploaded to canvas as a *pdf*. Make sure that if you scan your handwritten notes that they are legible and appropriately oriented. If you use an online resource to solve any problem, please appropriately cite that source.

Problem 1. (Lipschitz.) One of the key assumptions required for existence and uniqueness of solutions to differential equations is the Lipschitz condition on the dynamics with respect to the state-variable argument—i.e., we need the dynamics to be sufficiently regular. Consider the following systems of differential equations:

$$\mathcal{S}_1 = \begin{cases} \dot{x}_1 &= -x_1 + e^t \cos(x_1 - x_2) \\ \dot{x}_2 &= -x_2 + 15\sin(x_1 - x_2) \end{cases}$$

$$\mathcal{S}_2 = \begin{cases} \dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= -x_2 \end{cases}$$

a. Do they satisfy a global Lipschitz condition? Why? Provide a formal proof or counter example.

b. Let $\phi(t)$ be the solution of $\mathcal{S}_2$ due to the initial condition x_0 at $t_0 = 0$. Your friend says that ϕ is uniquely defined on $\mathbb{R}_+$ and for all $x_0 \in \mathbb{R}^2$, $\phi(t) \to 0$ as $t \to \infty$. Do you agree or disagree? Why? Provide a formal justification for your answer.

Solution.

a. For the first system, we have

$$f(x,t) = \begin{bmatrix} -x_1 + e^t \cos(x_1 - x_2) \\ -x_2 + 15\sin(x_1 - x_2) \end{bmatrix}$$

Recall from **[510]** that all finite dimensional norms are "equivalent". Using the norm $\|\cdot\|_\infty$ (since it is convenient), we have

$$\|f(x,t) - f(y,t)\| \le \|Df\|\|x-y\| \le \max\{1 + 2e^t, 31\}\|x-y\|$$

For the second system, we claim that it is not globally Lipschitz. Indeed, we only need to find an x and y such that there is no constant k such that

$$\|f(x) - f(y)\| \le k\|x-y\|$$

where

$$f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ -x_2 \end{bmatrix}$$

Consider

$$x = \begin{bmatrix} a \\ a \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then,

$$\left| f\begin{bmatrix} a \\ a \end{bmatrix} - f\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right| = \left| \begin{bmatrix} x(-1+x) \\ x \end{bmatrix} \right| = \left| \begin{bmatrix} x \\ x \end{bmatrix} \right| \left| \begin{bmatrix} x-1 \\ 1 \end{bmatrix} \right| > k \left| \begin{bmatrix} x \\ x \end{bmatrix} \right| = k \left| \begin{bmatrix} x \\ x \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right|$$

For any $k > 0$, we can find a vector $\begin{bmatrix} a \\ a \end{bmatrix}$ such that the above holds.

b. Agree. The second ODE for x_2 does not depend on x_1 and it satisfies the conditions of the ODE theorem. Hence, it has a solution $x_2(t) = e^{-t}x_2(0)$. We can plug this into the ODE $\dot{x}_1 = -x_1 + x_1x_2$ for x_2. Then, we can see that the resulting ODE satisfies the conditions of the ODE theorem. Hence, it has a solution given by $x_1(t) = x_1(0)\exp(-t + x_2(0)(\exp(-t) - 1))$. It is easy to see that $x_i(t) \to 0$ as $t \to \infty$ for all initial conditions and for each $i \in \{1, 2\}$.

Problem 2.(Perturbed nonlinear systems.) Suppose that some physical system obeys the differential equation

$$\dot{x} = f(x, t),\ x(t_0) = x_0,\ \forall t \geq t_0$$

where $f(\cdot, \cdot)$ obeys the conditions of the fundamental theorem. Suppose that as a result of some perturbation the equation becomes

$$\dot{z}(t) = f(z, t) + g(t),\ z(t_0) = x_0 + \delta x_0,\ \forall t \geq t_0$$

Given that for $t \in [t_0, t_0 + T]$, $\|g(t)\| \leq \varepsilon_1$ and $\|\delta x_0\| \leq \varepsilon_0$, find a bound on $\|x(t) - z(t)\|$ valid on $[t_0, t_0 + T]$.
Hint: Compare the norm of the difference of the two solutions, and revisit the proof of existence and uniqueness of ODEs provided in the lecture notes (or Appendix B, **[C&D]**).

Solution. Consider the differential equations

$$\dot{x} = f(x, t), \quad x(t_0) = x_0,$$

and

$$\dot{z} = f(z, t) + g(t), \quad z(t_0) = x_0 + \delta x_0$$

where $f(\cdot, \cdot)$ satisfies the conditions of the fundamental theorem, $\|g(t)\| \leq \varepsilon_1$, and $\|\delta x_0\| \leq \varepsilon_0$. We want to find a bound on $\|x(t) - z(t)\|$ that is valid in $t \in [t_0, t_0 + T]$.
Consider the difference in the two differential equations:

$$\dot{z} - \dot{x} = f(z, t) + g(t) - f(x, t), \quad z(t_0) - x(t_0) = \delta x_0.$$

Then, integrating to find the solution and evaluating the norm, we have

$$\begin{aligned}
\|z(t) - x(t)\| &= \| \int_{t_0}^{t} f(z, \tau) - f(x, \tau) + g(\tau) d\tau + \delta x_0 \| \\
&\leq \int_{t_0}^{t} (\|f(z, \tau) - f(x, \tau)\| + \|g(\tau)\|) d\tau + \|\delta x_0\| \\
&\leq \int_{t_0}^{t} (\|f(z, \tau) - f(x, \tau)\| + \varepsilon_1) d\tau + \varepsilon_0
\end{aligned}$$

Applying the Lipschitz continuity of $f(\cdot, \cdot)$ and taking the supremum of the Lipschitz function $k(t)$ on the interval $[t_0, t_0 + T]$, we have

$$\|z(t) - x(t)\| \leq \int_{t_0}^{t} \left(\sup_{\tau \in [t_0, t_0 + T]} k(\tau) \|z(\tau) - x(\tau)\| + \varepsilon_1 \right) d\tau + \varepsilon_0 \tag{1}$$

For the sake of simplicity of notation, define $C = \sup_{\tau \in [t_0, t_0 + T]} k(\tau)$. Now, we want to rewrite the above inequality so that we can apply the Bellman-Gronwall lemma which will give us a bound on the LHS.

$$\|z(t) - x(t)\| + \frac{\varepsilon_1}{C} \leq \int_{t_0}^{t} C \left(\|z(\tau) - x(\tau)\| + \frac{\varepsilon_1}{C} \right) d\tau + \varepsilon_0 + \frac{\varepsilon_1}{C} \tag{2}$$

So, by the Bellman-Gronwall lemma, we have

$$\|z(t) - x(t)\| + \frac{\varepsilon_1}{C} \leq \left(\varepsilon_0 + \frac{\varepsilon_1}{C}\right) \exp\left[\int_{t_0}^{t} C d\tau\right] \tag{3}$$

Thus,

$$\|z(t) - x(t)\| \leq \left(\varepsilon_0 + \frac{\varepsilon_1}{C}\right) \exp\left[\int_{t_0}^{t} C d\tau\right] - \frac{\varepsilon_1}{C} \tag{4}$$

for all $t \in [t_0, t_0 + T]$.

Problem 3. (State transition matrix properties.)

a. For nonsingular $M(t) \in \mathbb{R}^{n\times n}$, determine an expression for

$$\frac{d}{dt} M^{-1}(t)$$

b. Now, using part a. find and expression for

$$\frac{d}{d\tau}\Phi(t,\tau)$$

where $\Phi(t,\tau)$ is the state transition matrix of $\dot{x} = A(t)x$. Hint: state transition matrices are non-singular and hence, invertible.

Note: while this problem appears abstract, we will actually see that this expression for ODE $\frac{d}{d\tau}\Phi(t,\tau)$ (i.e., the derivative is with respect to the second argument) is important for understanding not only stablity (see next problem) but also for understanding how to run the ODE in reverse time, and in particular is important for connecting reachability (from the origin) and controllability (to the origin) which are part of **[Module 3]**.

Solution.

a. Since $M(t)$ is non-singular, it is invertible so that

$$M^{-1}(t)M(t) = I, \ \forall t$$

Hence,

$$\begin{aligned} \dot{M}^{-1}(t)M(t) + M^{-1}(t)\dot{M}(t) = 0 \implies & \dot{M}^{-1}(t)M(t) = -M^{-1}(t)\dot{M}(t) \\ \implies & \dot{M}^{-1}(t) = -M^{-1}(t)\dot{M}(t)M^{-1}(t) \end{aligned}$$

b. First, state transition matrices are non-singular so that

$$\Phi^{-1}(t,\tau) = \Phi(\tau,t)$$

Using the above,

$$\begin{aligned} \frac{d}{d\tau}\Phi(t,\tau) = \frac{d}{d\tau}\Phi^{-1}(\tau,t) &= -\Phi^{-1}(\tau,t)\dot{\Phi}(\tau,t)\Phi(t,\tau) \\ &= -\Phi^{-1}(\tau,t)A(\tau)\Phi(\tau,t)\Phi^{-1}(\tau,t) \\ &= -\Phi^{-1}(\tau,t)A(\tau) \\ &= -\Phi(t,\tau)A(\tau) \end{aligned}$$

Problem 4.(Matrix Differential Equations and Lyapunov Stability.) Let $\dot{x} = A(t)x$ be exponentially stable, and define

$$P(t) = \int_t^\infty \Phi(\tau,t)^*\Phi(\tau,t)\ d\tau,\ \forall t \geq 0$$

where $\Phi(t,t_0)$ is the state transition matrix for $\dot{x} = A(t)x$. Exponential stability is needed for $P(t)$ to be well defined. Use the previous problem to show that this $P(t)$ is in fact the solution to the matrix differential equation

$$\dot{P}(t) = -A(t)^*P(t) - P(t)A(t) - I,$$

and show that $P(t)^* = P(t)$.

Note. Why is this important? What this shows is that the time derivative of

$$v(x,t) = x^\top P(t)x$$

is decreasing along trajectories. Indeed,

$$\dot{v}(x,t) = \dot{x}^\top P(t)x + x^\top \dot{P}(t)x + x^\top P(t)\dot{x} = x^\top (A(t)^\top P(t) + P(t)A(t) + \dot{P}(t))x = -x^\top x \leq 0$$

And hence, as long as $v(t,x)$ is a positive definite function (which it is when $A(t)$ is stable see **[C&D]** Lemma 146 Chapter 7), then $v(t,x)$ is a Lyapunov function for the system.

Solution. We will use out trick of the Fundamental Theorem of ODEs. Indeed, we have that

$$\begin{aligned}
\frac{d}{dt}\left(\int_t^\infty \Phi(\tau,t)^*\Phi(\tau,t)\ d\tau\right) &= -\Phi(t,t)^*\Phi(t,t) + \int_t^\infty \frac{d}{dt}\left(\Phi(\tau,t)^*\Phi(\tau,t)\right)\ d\tau \\
&= -\Phi(t,t)^*\Phi(t,t) + \int_t^\infty \left(-\Phi(\tau,t)A(t))^*\Phi(\tau,t) - \Phi(\tau,t)^*\Phi(\tau,t)A(t)\right)\ d\tau \\
&= -I - A^*(t)\left(\int_t^\infty \Phi(\tau,t)^*\Phi(\tau,t)\ d\tau\right) - \left(\int_t^\infty \Phi(\tau,t)^*\Phi(\tau,t)\ d\tau\right)A(t) \\
&= -I - A^*(t)P(t) - P(t)A(t)
\end{aligned}$$

as claimed.
Showing that $P(t)^* = P(t)$ is straightforward:

$$\left(\int_t^\infty \Phi(\tau,t)^*\Phi(\tau,t)\ d\tau\right)^* = \left(\int_t^\infty (\Phi(\tau,t)^*\Phi(\tau,t))^*\ d\tau\right) = \left(\int_t^\infty \Phi(\tau,t)^*\Phi(\tau,t)\ d\tau\right).$$

Problem 5. (T-Periodic Systems: Existence.) Consider the linear system

$$\dot{x} = Ax + w(t)$$

where $w(t)$ is T periodic—i.e. $w(t+T) = w(t)$ and in particular $w(0) = w(T)$. There exists a T–periodic solution to this system, meaning there exists an $x(0)$ such that $x(t+T) = x(t)$. Find this solution by first determining $x(0)$.

Solution. We know that

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}w(\tau)\ d\tau$$

To make $x(t)$ T–periodic, we first require $x(0) = x(T)$. Hence,

$$x(T) = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}w(\tau)\ d\tau = x_0$$

so that

$$x_0 = (I - e^{AT})^{-1}\int_0^T e^{A(T-\tau)}w(\tau)\ d\tau$$

Now, we claim that

$$x(0) = x(T) \implies x(t) = x(t+T)$$

proof of claim: Denote $f(t) = x(t+T)$. Then

$$\begin{aligned} \dot{f}(t) &= Ax(t+T) + w(t+T) = Af(t) + w(t) \\ f(0) &= x(T) = x(0) \end{aligned}$$

Therefore, from the Fundamental Theorem of ODEs, $f(t) = x(t+T) = x(t)$.

Problem 6. (Properties of State Transition Matrices.) Consider the differential equation

$$\dot{x}(t) = (A(t) + B(t))x(t), \tag{5}$$

where $A(\cdot), B(\cdot) \in \mathrm{PC}(\mathbb{R}_+, \mathbb{R}^{n\times n})$. Let $\Phi_A(t, t_0)$ be the state transition matrix corresponding to

$$\dot{x} = A(t)x, \quad x(t_0) = x_0$$

and define

$$M(t) = \Phi_A(0, t)B(t)\Phi_A(t, 0).$$

Show that the state transition matrix of (5) is of the form

$$\Phi_A(t, 0)\Phi_M(t, t_0)\Phi_A(0, t_0)$$

where Φ_M is the state transition matrix corresponding to $\dot{z}(t) = M(t)z(t)$.
Hint: We showed this same result in class but for the linear time invariant case.

Solution.
general solution: Need to show that

$$\Phi_A(t, 0)\Phi_M(t, t_0)\Phi_A(0, t_0)$$

solves the ODE

$$\dot{X} = (A + B)X$$

$$\begin{aligned} \frac{d}{dt}\Phi_A(t,0)\Phi_M(t,t_0)\Phi_A(0,t_0) &= \dot{\Phi}_A(t,0)\Phi_M(t,t_0)\Phi_A(0,t_0) + \Phi_A(t,0)\dot{\Phi}_M(t,t_0)\Phi_A(0,t_0) \\ &= A\Phi_A(t,0)\Phi_M(t,t_0)\Phi_A(0,t_0) + \Phi_A(t,0)M(t)\Phi_M(t,t_0)\Phi_A(0,t_0) \\ &= A\Phi_A(t,0)\Phi_M(t,t_0)\Phi_A(0,t_0) + \Phi_A(t,0)\Phi_A(0,t)B\Phi_A(t,0)\Phi_M(t,t_0)\Phi_A(0,t_0) \\ &= A\Phi_A(t,0)\Phi_M(t,t_0)\Phi_A(0,t_0) + B\Phi_A(t,0)\Phi_M(t,t_0)\Phi_A(0,t_0) \\ &= (A+B)\Phi_A(t,0)\Phi_M(t,t_0)\Phi_A(0,t_0) \end{aligned}$$