EE445 Mod4-Section2: Convexity II

References: [Optimization Models: Calafiore & El Ghaoui] Chapter 8

Convex Hull

From Wikipedia:

> *The convex hull may be defined either as the intersection of all convex sets containing a given subset of a Euclidean space.*
>
> *Equivalently as the set of all convex combinations of points in the subset.*
>
> *For a bounded subset of the plane, the convex hull may be visualized as the shape enclosed by a rubber band stretched around the subset.*

Exercise 1 - Convex Hull

Question What is the convex hull (in $\mathbf{R}^2$) of the points $e_1, e_2, -e_1, -e_2$ (where e_i is the ith coordinate vector)? Draw a picture and name the set.

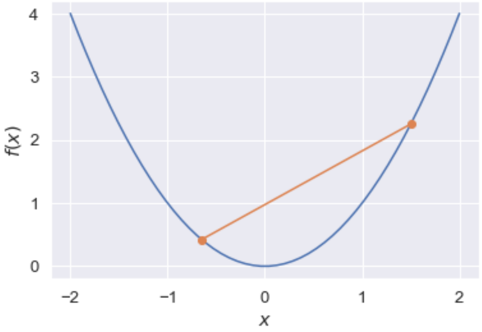
Convex functions

From last lecture: $f : \mathbf{R}^n \mapsto \mathbf{R}$ is a **convex function** if

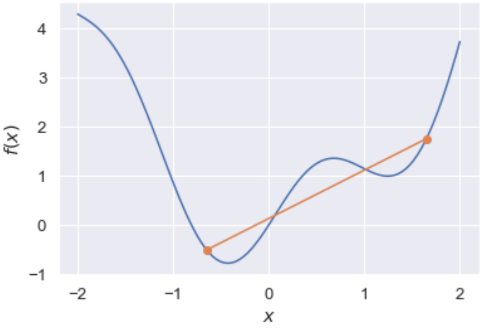
$$f\Big(\lambda x + (1-\lambda)y\Big) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in \mathbf{R}^n$ and all $0 \leq \lambda \leq 1$.

Convex



Nonconvex



1st and 2nd order convexity condition

1st-order condition: differentiable f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \mathbf{dom}\, f$$

2nd-order conditions: for twice differentiable f with convex domain f is convex if and only if

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \mathbf{dom}\, f$$

Tools to show convexity

practical methods for establishing convexity of a function

1. verify the definition: show for all $x, y \in \mathbf{dom}\, f$ and all $0 \leq \lambda \leq 1$,

$$f\Big(\lambda x + (1-\lambda) y\Big) \leq \lambda f(x) + (1-\lambda) f(y)$$

2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$
3. show that f is obtained from simple convex functions by operations that preserve convexity:
   - nonnegative weighted sum
   - composition with affine function
   - pointwise maximum
   - partial minimization

Properties that preserve convexity

Positive Weighted Sum: αf is convex if f is convex, $\alpha \geq 0$
$f_1 + f_2$ convex if f_1, f_2 convex

Composition with affine function: Consider the affine function $x \mapsto Ax + b$, with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$,
then the function $g(x) = f(Ax + b)$ is a convex function if f is convex.

Pointwise maximum If f_1, ..., f_m are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex.

Partial minimization if $f(x, y)$ is convex in (x, y) (note that this means jointly convex in the variables) and C is a convex set, then

$$g(x) = \min_{y \in C} f(x, y)$$

is also convex

Exercise 2 - Minimum-weight path in a graph

Consider a weighted graph $G = (V, E)$ where V is the set of vertices and E the set of edges, and each edge i has a weight $w_i \geq 0$. Then the weight of a path from vertex $s \in V$ to $t \in V$ is equal to the sum of the weights of the edges along that path. Show that the weight of the *minimum-weight path* between vertices s and t is a *concave* function of the weights w_i.

Exercise 3

Question Show that the sum of the r largest entries of a vector x is convex.