EE445 Mod4-Lec1: Convexity and Convex Sets

References: [Optimization Models: Calafiore & El Ghaoui] Chapter 8

Optimization: Overview

A general optimization problem has the form

$$\begin{array}{ll} \text{minimize}_x & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

with components

- $x = (x_1, \ldots, x_n)$ - optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ - objective function
- $f_i : \mathbf{R}^n \to \mathbf{R}$ - constraint functions; b_i - constraint bounds

Optimization: Problem Classes

An important class: **convex optimization** problems

“With only a bit of exaggeration, we can say that if you formulate a practical problem as a convex optimization problem, then you have solved the original problem."

Quote from: S. Boyd and L. Vandenberghe. Convex Optimization. Cambridge University Press, 2004.

Convex sets

$S \subseteq \mathbf{R}^n$ is a **convex set** if

$$x, y \in S, \;\; \lambda, \mu \geq 0, \;\; \lambda + \mu = 1 \Longrightarrow \lambda x + \mu y \in S$$

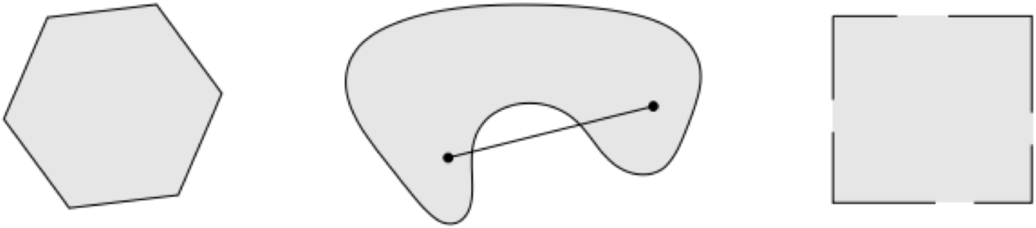
or equivalently,

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta) x_2 \in C$$

geometrically: $x, y \in S \Rightarrow$ segment $[x, y] \subseteq S$
Examples:

1. Hyperplanes and halfspaces
2. Euclidean balls
3. Ellipsoids

Exercise 1



Exercise 2

Question Show that the intersection of convex sets is convex.

Showing convexity using properties: Exercise 3

A *wedge*, that is, $\{x \in \mathbf{R}^n \mid a^T x \leq b,\ c^T x \leq d\}$.

Showing convexity using properties: Exercise 4

The set of points closer (in Euclidean norm) to a point a than a point b, i.e., $\{x \in \mathbf{R}^n \mid \|x - a\|_2 \leq \|x - b\|_2\}$.

Exercise 5

Question Consider the set defined by the following inequalities

$$(x_1 \geq x_2 - 1 \text{ and } x_2 \geq 0) \text{ or } (x_1 \leq x_2 - 1 \text{ and } x_2 \leq 0)$$

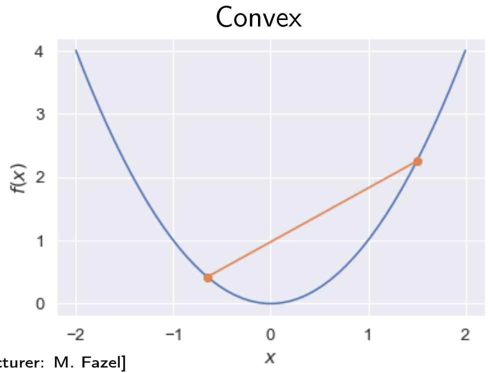
Is the set convex?

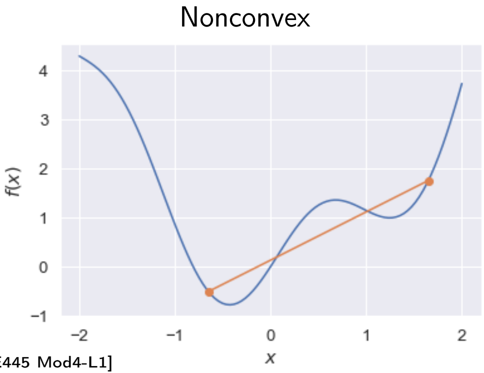
Convex functions

From last lecture: $f : \mathbf{R}^n \mapsto \mathbf{R}$ is a **convex function** if

$$f\Big(\lambda x + (1-\lambda)y\Big) \le \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in \mathbf{R}^n$ and all $0 \le \lambda \le 1$.





Exercise 6: Convexity using definition

Question

$$f(x) = \|Ax - b\|_2$$

Exercise 7: Quadratic-over-linear:

Question $f(x, y) = x^2/y$ *Hint: Check using the Hessian condition.*

Quad-over-linear: Solution

$$\nabla^2 f(x,y) = \frac{2}{y^3}\begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} = \frac{2}{y^3}\begin{bmatrix} y \\ -x \end{bmatrix}\begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0$$

convex for $y > 0$.