EE445 Mod3-Lec2: Supplement

References:

- [CE-OptMod]: Chapter: 5

Common Matrix Norms

Other norms of interest include the 1-norm and ∞-norms.

- 1-**norm** (ℓ_1): consider $x \in \mathbb{R}^n$. The ℓ_1-norm is given by $\|x\|_1 = \sum_{i=1}^n |x_i|$
- ∞-**norm** (ℓ_∞): consider $x \in \mathbb{R}^n$. The ℓ_∞-norm is given by $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$

We can define induced norms from these ℓ_p norms:

$$\|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_1 = \max_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}| \quad \text{i.e., the max column sum}$$

$$\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}| \quad \text{i.e., the max row sum,}$$

Proof of $\|A\|_\infty$ being max row sum

- Indeed, we have that

$$\|Ax\|_\infty = \max_{i=1,\ldots,m} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \le \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}||x_j| \le \max_{j=1,\ldots,n} |x_j| \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|.$$

- Then since $\|x\|_\infty = 1$, we have that

$$\|Ax\|_\infty \le \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|$$

- Next, you show that there exists $\tilde{x}$ such that $\|\tilde{x}\|_\infty = 1$ and

$$\|A\tilde{x}\|_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|$$

Proof of $\|A\|_\infty$ being max row sum

- Next, you show that there exists $\tilde{x}$ such that $\|\tilde{x}\|_\infty = 1$ and

$$\|A\tilde{x}\|_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|$$

- Indeed, the $\|\tilde{x}\|_\infty = 1$ that maximizes this quantity is $\tilde{x}$ such that its entries are $\tilde{x}_j = \text{sign}(a_{i^*j})$ for $j = 1, \ldots, n$ where $i^* = \arg\max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|$.
- That is, with this choice of x, the inequalities above become equalities.