EE445 Mod2-Lec3: Least Squares Classification

References:

- [VMLS]: Chapter 14

Outline

- What is classification?
- Different error rates
- least squares classifier
- Multi-class classifiers

Binary Classification with Least Squares

Classification

- **M2-L2**: goal was to predict an outcome y from some data x
- **M2-L3** (Classification): the outcome y takes on only a finite number of values, and hence is sometimes called a **label**, or in statistics, a **categorical**.
- Example [Binary Classification]: $y \in \{-1, 1\}$ or $y \in \{0, 1\}$ $y \in \{\text{`True'}, \text{`False'}\}$
- Relationship: $\hat{y} = f(x)$ where $f : \mathbb{R}^n \to \{-1, +1\}$
- Classifier: f is called the **classifier** since it takes in vectors $x \in \mathbb{R}^n$ and classifies them as either $f(x) = +1$ or $f(x) = -1$.

Classification Examples

- **Email spam detection.**
  - *Feature vector*: $x \in \mathbb{R}^n$ contains features of an email message like word counts etc.
  - *Outcome*: $y = +1$ if an email represented by feature vector x is `SPAM` and -1 otherwise.
- **Fraud detection.**
  - *Feature vector*: $x \in \mathbb{R}^n$ contains features associated with a credit card user such as average monthly spending, median prices of purchases over last week, etc.
  - *Outcome*: $y = +1$ for `fradulent transactions`, and -1 otherwise.
- **Document Classification.**
  - *Feature vector*: $x \in \mathbb{R}^n$ is a word count (or histogram) vector for a document
  - *Outcome*: $y = +1$ if the document has some topic (e.g., politics) and -1 otherwise

Prediction Errors

- For a given data point (x, y) with predicted outcome $\hat{y} = f(x)$, there are four possibilities:
  1. *True Positive*: $y = +1$ and $\hat{y} = +1$ [correct prediction]
  2. *True Negative*: $y = -1$ and $\hat{y} = -1$ [correct prediction]
  3. *False Positive*: $y = -1$ and $\hat{y} = +1$ [incorrect prediction, type I error]
  4. *False Negative*: $y = +1$ and $\hat{y} = -1$ [incorrect prediction, type II error]

Error Rates

Consider data set $(x^(1), \ldots, x^{(N)}), \quad (y^{(1)}, \ldots, y^{(N)})$ and model f.

- *Error rate*:

- *True positive rate* (sensitivity/recall rate):

- *False positive rate* (false alarm rate):

- *True negative rate* (specificity):

- *Precision*:

Confusion Matrix

- *Good classifier*: small (near zero) error rate and false positive rate, and high (near one) true positive rate, true negative rate, and precision.
- Which of these metrics is more important depends on the particular application.

	prediction		
outcome	$\hat{y} = +1$	$\hat{y} = -1$	total
$y = +1$	$N_{\mathtt{tp}}$	$N_{\mathtt{fn}}$	$N_{\mathtt{p}}$
$y = -1$	$N_{\mathtt{fp}}$	$N_{\mathtt{tn}}$	$N_{\mathtt{n}}$
all	$N_{\mathtt{tp}} + N_{\mathtt{fp}}$	$N_{\mathtt{fn}} + N_{\mathtt{tp}}$	N

Least Squares Classifier

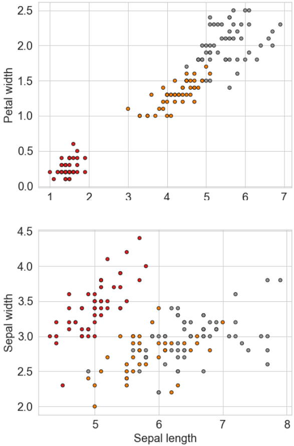
- Note: sophisticated methods exist for constructing binary classifiers—e.g., logistic regression and support vector machines—which are beyond this lecture.
- Least squares classifier: this is a simple method that works well in many cases
- Process:
  - do ordinary real-valued least squares fitting of the outcome, ignoring that $y \in \{-1, +1\}$
  - i.e.,
  - **final classifier (least squares classifier)**:

Intuition for Least Squares Classifier

- The value $\tilde{f}(x)$ is a number "near" $+1$ when $y = +1$ and near -1 when $y = -1$
- Forced to guess one of the two possible outcomes, $\text{sign}(\tilde{f}(x))$ is a good choice—it is the nearest neighbor of $\tilde{f}(x)$ among $\{-1, +1\}$
- $\tilde{f}(x)$ also tells us our confidence in our assignment

Example

- Iris data set: classical ML data set
- Three types of iris:
  - `setosa`, `versicolour`, `virginica`
- Four features:
  - x_1 `sepal length [cm]`, x_2 `sepal width [cm]`
  - x_3 `petal length [cm]`, x_4 `petal width [cm]`
- 50 samples of each type
- Goal: build classifier to detect if iris is `virginica` or not



Iris Data Set Example: Confusion Matrix

- data matrix

$$A = \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & x_4^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_1^{(N)} & x_2^{(N)} & x_3^{(N)} & x_4^{(N)} \end{bmatrix}$$

- labels: $y = (\underbrace{-1, \dots, -1}_{100 \text{ times}}, \underbrace{+1, \dots, +1}_{50 \text{ times}})$
- $\min_x \|Ax - y\|_2^2$
- **Solution**:

Iris Data Set Example: Confusion Matrix

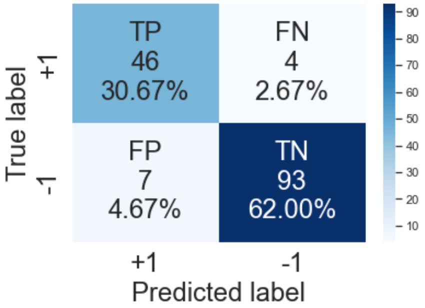
- Precision:

$$\frac{N_{\mathtt{tp}}}{N_{\mathtt{tp}} + N_{\mathtt{fp}}}$$

- Accuracy:

$$\frac{N_{\mathtt{tp}} + N_{\mathtt{tn}}}{N}$$

- F1-score is the harmonic mean of precision (P) and recall (R): $\frac{2PR}{P+R}$
  - recall: $N_{\mathtt{tp}}/(N_{\mathtt{tp}} + N_{\mathtt{fn}})$
  - want it near 1



Accuracy=0.927; Precision=0.959
Recall=0.930; F1 Score=0.944

Cross validation

- Just like in the last lecture, we can use cross validation to our least squares classifier.
- see `Mod2-Lec3.ipynb` for example

Receiver Operating Characteristic [ROC] Curves

Modified Classifier with Skewed Decision Boundary

- Modified least squares classifier: skew the decision boundary

$$f(x) = \mathrm{sign}(\tilde{f}(x) - \alpha) = \begin{cases} +1, & \tilde{f}(x) \geq \alpha \\ -1, & \tilde{f}(x) < \alpha \end{cases}$$

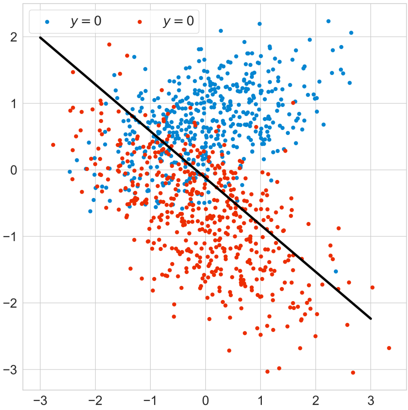
- $\alpha > 0$: the guess $f(x) = +1$ is less frequent $\implies$
  - the numbers in the first column (TP, FP) of the confusion matrix go down, and the numbers in the second column (FN,TN) go up
  - i.e., $\alpha > 0 \implies \mathtt{FPR}\uparrow$ which is **good**, yet $\mathtt{TPR}\downarrow$ which is **bad**
  - Note: sum of the numbers in each row is always the same
- $\alpha < 0$: the guess $f(x) = +1$ is more frequent
  - $\implies \mathtt{TPR}\uparrow$ which is **good**, yet $\mathtt{FPR}\downarrow$ which is **bad**
- We choose the decision threshold α depending on how much we care about these different metrics in the application

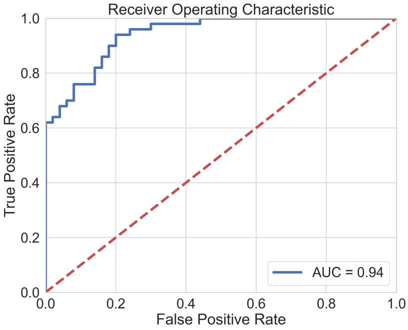
Receiver Operating Characteristic [ROC] Curves

- By sweeping α over a range, we obtain a family of classifiers that vary in their true positive and false positive rates
- Two plots of interest:
  1. the false positive and negative rates, as well as the error rate, as a function of α
  2. `[ROC]`: true positive rate on the y-axis and false positive rate on the x-axis [More Common to Plot]
- **Cool History Fact:** The name comes from radar systems deployed during World War II, where $y = +1$ means that an enemy vehicle (or ship or airplane) is present, and $\hat{y} = +1$ means that an enemy vehicle is detected.

Example: `Mod2-Lec3.ipynb, example 3`

- Randomly generated binary classification problem: $m = 1000$





Example: `Mod2-Lec3.ipynb, example 3`

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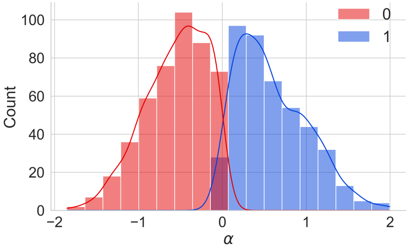
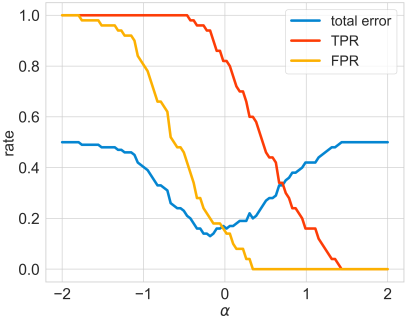




Multi-Class Classification with Least Squares

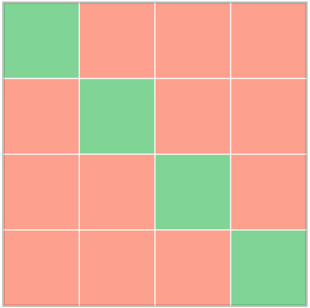
Multi-Class Classifiers

- K Class Classification: # of labels is greater than two ($K > 2$)
  - e.g., Likert scale labels: "Strongly Disagree", "Disagree", "Neutral", "Agree", "Strongly Agree"
  - e.g., Iris data set: three types of iris `setosa, versicolour, virginica`
- **Def.** A multi-class classifier is a function $f : \mathbb{R}^n \to \{1, \ldots, K\}$
  - Given a feature vector x, the classifier f returns $f(x) \in \{1, \ldots, K\}$
- Examples
  - Handwritten digit classification (MNIST)
  - Marketing demographic classification—e.g., "college-educated women aged 25–30", "men without college degrees aged 45–55"
  - Disease diagnosis: K possible outcomes for disease
  - Document topic prediction: K possible topics
  - Detection in communications: translate message into K possible signals

Confusion Matrix

- For a multi-class classifier f and a given data point (x, y), with predicted outcome $\hat{y} = f(x)$, there are K^2 possibilities corresponding to all the pairs of values of y, and $\hat{y}$.

- Confusion matrix C: for a given training or test data set with N elements, the numbers of K^2 occurences are arranged into a $K \times K$ matrix where C_{ij} is the number of data points for which $y = i$ and $\hat{y} = j$
- Diagonal of C contains the number of cases for which the prediction is correct



Measures for Prediction Error

- When $K = 2$ we have two types of errors: false positives, false negatives
- More complicated when $K > 2$: From the entries of the confusion matrix we can derive various measures of the accuracy of the predictions

Overall Error Rate

- **Overall Error Rate**:

- This measure implicitly assumes that all errors are equally bad.
- In many applications this is not the case; e.g., some medical misdiagnoses might be worse for a patient than others.

True Label Rate

- **True Label Rate for Class i:**

Least Squares Multi-Class Classifier

- The idea behind the least squares Boolean classifier can be extended to handle multi-class classification problems
- *one-vs-others* or *one-vs-all*: for each possible label value, construct a new data set with the Boolean label $+1$ if the label has the given value, and -1 otherwise.
- Select the one with the highest level of confidence (i.e., best least squares fit):

Example

- consider a multi-class classification problem with 3 labels.
- construct 3 different least squares classifiers: a) 1 versus $\{2 \text{ or } 3\}$, b) 2 versus $\{1 \text{ or } 3\}$, and c) 3 versus $\{1 \text{ or } 2\}$.
- e.g., suppose we have

$$\tilde{f}_1(x) = -0.7, \quad \tilde{f}_2(x) = +0.2, \quad \tilde{f}_3(x) = +0.8$$

- $f(x) = 3$ since $\tilde{f}_3(x)$ is larger than $\tilde{f}_1(x)$ and $\tilde{f}_2(x)$