EE445 Mod1-Lec3: Linear Algebra III

References:

- [VMLS]: Chapter 5, 6, 7

Gram-Schmidt (orthogonalization) algorithm

- an algorithm to check if $a_1, \ldots, a_k$ are linearly independent
- we'll see later it has many other uses
- useful properties:
  - suppose you've orthogonalized vectors $a_1, \ldots, a_k$, and a new vector a_{k+1} is then added to the list. G-S lets you update the previous solution easily (and efficiently).
  - an "incremental" algorithm that handles new data arriving—related to "online" learning

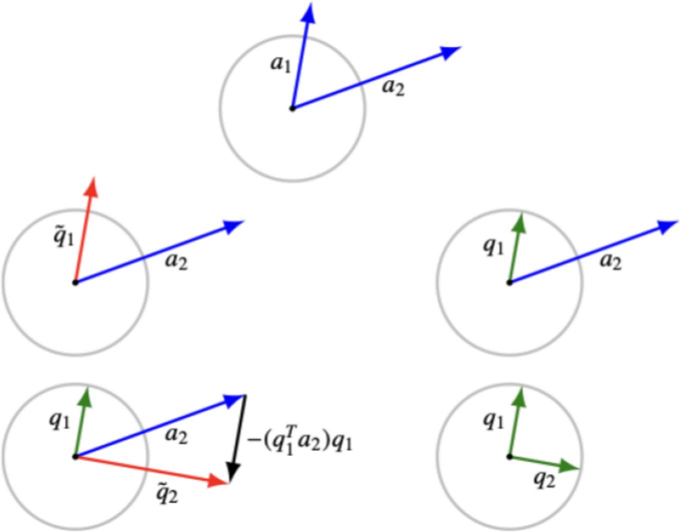
Gram-Schmidt algorithm

given n-vectors $a_1, \ldots, a_k$
for $i = 1, \ldots, k$,

1. orthogonalization: $\tilde{q}_i = a_i - (q_1^T a_i)q_1 - \ldots - (q_{i-1}^T a_i)q_{i-1}$
2. test for lin. independence: if $\tilde{q}_i = 0$, quit
3. normalization: $q_i = \tilde{q}_i / \|\tilde{q}_i\|$

- if G-S stops early in iteration $i = j$, then a_j is a linear combination of $a_1, \ldots, a_{j-1}$ (so set of vectors is linearly dependent)
- if G-S doesn't stop early, then linearly independent

Example



Analysis of G-S algorithm

we show $q_1, \ldots, q_i$ are orthonormal, by induction

- assume it's true for $i-1$. orthogonalization step ensures

$$\tilde{q}_i \perp q_1, \ \ldots \ , \tilde{q}_i \perp q_{i-1}$$

- to see this, take inner product of both sides with q_j, $j < i$:

- normalization step ensures $\|q_i\| = 1$

Analysis of G-S algorithm

assume G–S has not terminated before step i: then

- a_i is a lin. comb. of $q_1, \ldots, q_i$:

$$a_i = \|\tilde{q}_i\| q_i + (q_1^T a_i) q_1 + \ldots + (q_{i-1}^T a_i) q_{i-1}$$

- q_i is also a lin. comb. of $a_1, \ldots, a_i$:
  if (by induction assumption) each $q_1, \ldots, q_{i-1}$ is a lin. comb. of $a_i, \ldots, a_{i-1}$, then for

$$q_i = \frac{1}{\|\tilde{q}_i\|} \left(a_i - (q_1^T a_i) q_1 - \ldots - (q_{i-1}^T a_i) q_{i-1} \right)$$

assume G-S terminates at step j: then a_j is a linear combination of $a_1, \ldots, a_{j-1}$

Analysis of G-S algorithm

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Review of Matrices

- a $m \times n$ matrix is a rectangular array of numbers, denoted as $A \in \mathbf{R}^{m \times n}$, e.g.,

$$\begin{bmatrix} 0 & 1 & -2.3 & 0.1 \\ 1.3 & 4 & -0.1 & 0 \\ 4.1 & -1 & 0 & 1.5 \end{bmatrix}$$

- A_{ij} is the i, jth element (entry); transpose: $(A^T)_{ij} = A_{ji}$
- shapes: tall ($m > n$), wide ($m < n$), square ($m = n$), diagonal, upper triangular,. . .
- column & row representation of matrix (a_i are column m-vectors, b_i are row n-vectors):

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}, \qquad A = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Examples

- *image:* X_{ij} is the pixel value in a gray-scale image
- *rainfall data:* X_{ij} is rainfall at location i on day j
- *feature matrix:* X_{ij} is value of feature i for entity j

a block matrix: $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$

ex: $B = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}$, $C = [-1]$, $D = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 5 \end{bmatrix}$, $E = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

Matrix Frobenius norm

- for $m \times n$ matrix A,

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^2 \right)^{1/2}$$

(in the book, F subscript is often dropped). agrees with vector norm if $n = 1$.

- satisfies norm properties:
  - $\|\alpha A\|_F = |\alpha| \|A\|_F$
  - $\|A + B\|_F \leq \|A\|_F + \|B\|_F$
  - $\|A\|_F \geq 0$; and $\|A\|_F = 0$ only if $A = 0$
- distance between two matrices: $\|A - B\|_F$
- (there are many other matrix norms, will see some later)

Examples

- **reversal matrix:** $f(x) = Ax = (x_n, \ldots, x_1)$
  Saw in HW0, P6.

- **running sum:** $f(x) = Ax = (x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, \sum_{i=1}^{n} x_i)$ with

Examples

- **centering matrix:** $\bar{x} = Ax$ is *centered* (de-meaned) version of x with

$$A = \begin{bmatrix} 1-1/n & -1/n & \dots & -1/n \\ -1/n & 1-1/n & \dots & -1/n \\ \vdots & & \ddots & \vdots \\ -1/n & -1/n & \dots & 1-1/n \end{bmatrix}$$

- **difference matrix** D and $y = Dx$ (vector of differences of consecutive entries of x):

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad Dx = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}$$

Matrix-vector product

- Define $y = Ax$, for $m \times n$ matrix A and n-vector x, as

$$y_i = A_{i1}x_1 + \ldots + A_{in}x_n, \quad i = 1, \ldots, m$$

- *row interpretation:*
  $y_i = b_i^T x$, $i = 1, \ldots, m$, where $b_1^T, \ldots, b_m^T$ are rows of A (so $y = Ax$ gives inner product of all rows of A with x)
- example: $A\mathbf{1} =$
- *column interpretation:*
  $y = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n$, where $a_1, \ldots, a_n$ are columns of A
- example: $Ae_j =$

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Ex: Feature matrix-weight vector

- $X = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$ is an $n \times N$ *feature matrix*
- column x_j is feature n-vector for object/example j
- X_{ij} is value of feature i for example j
- n-vector w is *weight* vector
- $s = X^T w$ is vector of *scores* for each example: $s_j = x_j^T w$

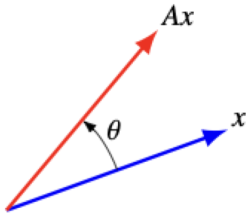
Ex: Input-output matrix

- consider $y = Ax$:
- n-vector x is *input* or action
- m-vector y is *output* or result
- A_{ij} is the *gain* from input j to output i
- e.g., if A is lower triangular, then y_i depends only on $x_1, \ldots, x_i$

Ex: Geometric transformations

- many geometric transformations and mappings of 2D and 3D vectors can be represented by $y = Ax$
- e.g., rotation by θ:

$$y = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} x$$



(to get the entries, look at Ae_1, Ae_2)

Ex: Incidence matrix in a graph

- graph with n vertices or nodes, m (directed) edges or links.
- incidence matrix is $n \times m$ matrix

$$A_{ij} = \begin{cases} 1, & \text{edge } j \text{ points to node } i \\ -1, & \text{edge } j \text{ points from node } i \\ 0 & \text{otherwise} \end{cases}$$

- ex with $n = 4$, $m = 5$:

$$A = \begin{bmatrix} -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Incidence matrix and flow conservation

- m-vector x gives flows (of something) along the edges
- examples: heat, money, power, mass, people,...
- $x_j > 0$ means flow follows edge direction
- Ax is n-vector that gives the total or net flows
- $(Ax)i$ is the net flow into node i
- $Ax = 0$ is *flow conservation*

Ex: Input-output convolution

- for n-vector a, m-vector b, the (discrete-time) convolution $c = a * b$ is the $(n+m-1)$-vector

$$c_k = \sum_{i+j=k+1} a_i b_j, \quad k = 1, \ldots, n+m-1$$

- as seen in ee341 (and ee235)
- e.g., with $n = 4$, $m = 3$:

$$\begin{aligned} c_1 &= a_1 b_1 \\ c_2 &= a_1 b_2 + a_2 b_1 \\ c_3 &= a_1 b_3 + a_2 b_2 + a_3 b_1 \\ &\vdots \\ c_6 &= a_4 b_3 \end{aligned}$$

Convolution and Toeplitz matrices

- can express $c = a * b$ using matrices as $c = T(b)a$, with the *Toeplitz* matrix

$$T(b) = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ b_2 & b_1 & 0 & 0 \\ b_3 & b_2 & b_1 & 0 \\ 0 & b_3 & b_2 & b_1 \\ 0 & 0 & b_3 & b_2 \\ 0 & 0 & 0 & b_3 \end{bmatrix}$$

Convolution example: moving average of time series

- n-vector x represents a time series
- convolution $y = a * x$ with $a = (1/3, 1/3, 1/3)$ is a 3-period *moving average*:

$$y_k = \frac{1}{3}(x_k + x_{k-1} + x_{k-2}), \quad k = 1, 2, \ldots, n+2$$

with x_k taken as zero for $k < 1$ and $k > n$.