EE445 Final review/practice

- course evals
- final exam : Fri 6/3 @ 8am, due Sat 6/4 @ noon

References:

- [VMLS]: Chapters 1-15 (except 9)
- [OM, Calafiore & El-Ghaoui]: See Module 3 refs, and
  Chapter 8, sections 8.1-8.3 (except 8.2.3)
  Chapter 13 (sections 13.1, 13.2, 13.3.1-5)

$\|Ax-b\|_2^2 = \sum_i (a_i^T x - b_i)^2$

HW6, Prob 2

Weighted least-squares cost as a function of weights. Let $a_1, \ldots, a_n \in \mathbf{R}^m$. In weighted LS, we minimize the objective $\sum_{i=1}^n w_i(a_i^T x - b_i)^2$ over $x \in \mathbf{R}^m$. Define the *optimal weighted least squares cost* as

$$g(w) = \min_x \sum_{i=1}^n w_i (a_i^T x - b_i)^2,$$

$w_i \geq 0$

with $\mathbf{dom}\, g = \{w \mid g(w) > -\infty\}$. Show that $g(w)$ is concave in w.

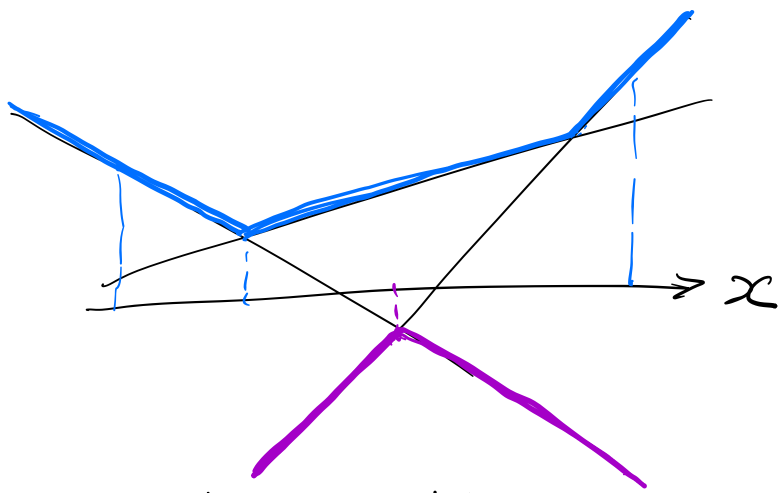
$$g(w) = \min_x \sum_{i=1}^n \underbrace{(a_i^T x - b_i)^2}_{c_{i,x}} w_i = \min_x \underbrace{\sum_{i=1}^n c_{i,x} w_i}_{c_x^T w} = \min_x (c_x^T w)$$

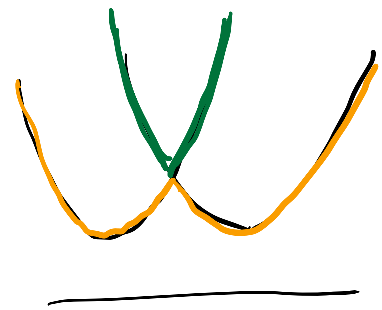
linear in w

for every fixed x, $c_x^T w$ is linear in w. $g(w)$ is pointwise min of linear functions, therefore concave.

recall: piecewise linear convex function:

$$f(x) = \max\{\underline{a_1^T x + b_1}, \underline{a_2^T x + b_2}, \underline{a_3^T x + b_3}\}$$





- how about pointwise min?

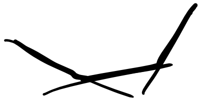
  pointwise min. of linear functions is concave

  " " convex fcts is " ?

  No.

- also true for infinite # of linear functions

HW6, Prob 3



Some measure of 'spread' of entries in a vector $x \in \mathbf{R}^n$:

1. $\phi_{\text{sprd}}(x) = \underbrace{\max_{i=1,\ldots,n} x_i}_{\text{convex}} - \underbrace{\min_{i=1,\ldots,n} x_i}_{\text{convex}}$

- $\max_i x_i = \max_i \underbrace{e_i^T x}_{\text{linear fct of } x}$ pointwise max of lines $\Rightarrow$ convex.

- $-\min_i x_i = \max_i (-x_i)$

& sum of 2 convex fcts is convex, so $\phi_{\text{sprd}}(x)$ is convex in x.

2. *standard deviation*, defined (recall Module 1, Lec. 2) as

$$\phi_{\text{stdev}}(x) = \left(\frac{1}{n} \underbrace{\sum_{i=1}^{n} x_i^2}_{x^T x} - \left(\frac{1}{n} \underbrace{\sum_{i=1}^{n} x_i}_{1^T x} \right)^2 \right)^{1/2}.$$

centered

$\tilde{x} = x - \text{ave}(x) 1$

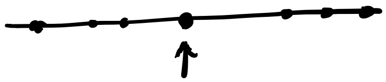
$$\phi_{\text{stdev}}(x) = \| \underbrace{\underbrace{\left(I - \tfrac{1}{n} 1 1^T\right)}_{A} x}_{Ax \text{ : linear in } x} \|_2$$

$$A = \begin{bmatrix} 1 & 1-1/n & \cdots & 1-1/n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$ centering

$\phi_{\text{stdev}}(x)$ is composition of $\|\cdot\|_2$ (that we know is convex) with affine map $x \to Ax$.

(this one was hard, totally fine if you didn't get it, or used the long bruteforce method of checking the definition of convexity – that works too)

3. *average absolute deviation from the median* of the values:



$$\phi_{\mathrm{aamd}}(x) = (1/n)\sum_{i=1}^{n} |x_i - \underline{\mathrm{med}(x)}|.$$

we use that:

$$\underbrace{\phi_{\mathrm{aamd}}(x) = \min_t \|x - t\mathbf{1}\|_1}$$

(we use this as "given" here. to prove it, have to argue that $t^* = \arg\min_t \|x - t\mathbf{1}\|_1 = \mathrm{med}(x)$, which needs extra work)

<u>partial min</u> property: $g(y) = \min_x f(x,y)$ convex if $f(x,y)$ is <u>jointly convex</u> in (x,y).

$f(x,t) = \|\underset{\smile}{x} - \underset{\smile}{t}\mathbf{1}\|_1$ is linear in both x, t, then composed with ℓ_1-norm (convex)

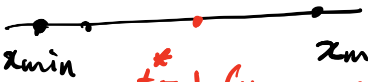
$\Rightarrow$ jointly convex in (x,t)

$\phi_{\mathrm{aamd}}(x) =$ partial min of $f(x,t) \Rightarrow$ convex

- $\phi(x) = \min_t \| x - t\mathbf{1} \|_1 \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R} \qquad \begin{bmatrix} x_1 - t \\ x_2 - t \\ \vdots \\ x_n - t \end{bmatrix}$

  $= \phi_{aamd}(x)$

- $\phi(x) = \min_t \| x - t\mathbf{1} \|_\infty = \min_t \left\| \begin{bmatrix} x_1 - t \\ \vdots \\ x_n - t \end{bmatrix} \right\|_\infty$



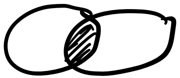
$t^* = \frac{1}{2}(x_{max} - x_{min})$

mid point

- $\phi(x) = \min_t \| x - t\mathbf{1} \|_2 = \min_t \left\| \begin{bmatrix} x_1 - t \\ \vdots \\ x_n - t \end{bmatrix} \right\|_2 \qquad t^* = \text{ave}(x) = \frac{1}{n} \mathbf{1}^T x$

[this slide covers more advanced material, just FYI, not for final]

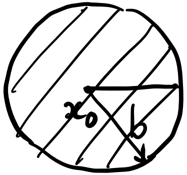
More convex sets

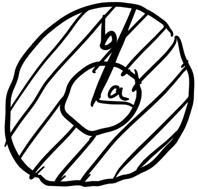


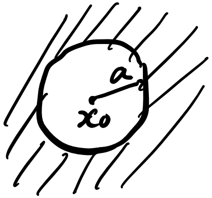
1. Is the set $\{x \in \mathbf{R}^n \mid a \leq \|x - x_0\|_2 \leq b\}$ with $b > 0$, convex when
   - $a > 0$? no.
   - $a = 0$? yes.

   Draw a (2D) picture.

$\{x \mid \|x-x_0\| \le b\} \cap \underbrace{\{x \mid \|x-x_0\| \ge a\}}$

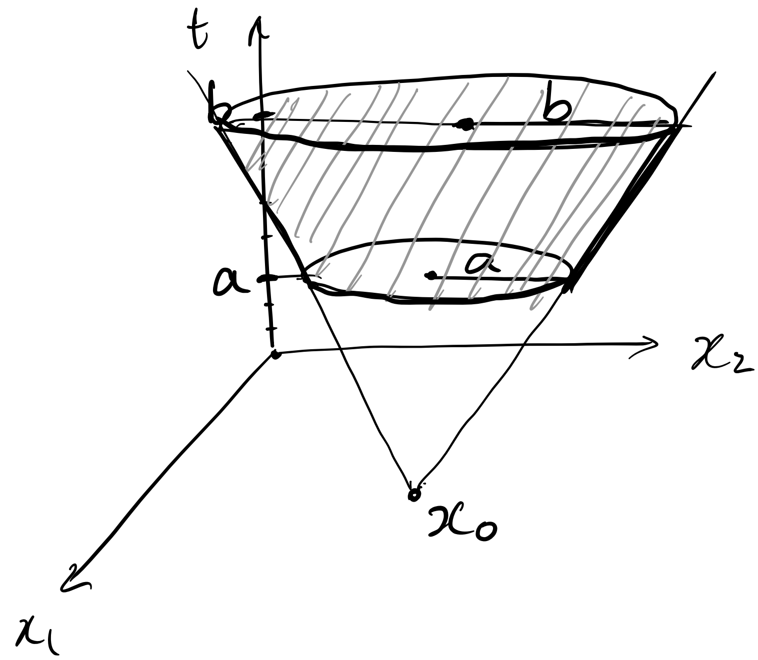






$x \in \mathbb{R}^n,\ t \in \mathbb{R}$

2. Is the set $\{(x, t) \in \mathbf{R}^{n+1} \mid \|x - x_0\|_2 \leq t,$ for all $a \leq t \leq b\}$ convex for $a > 0$? Draw a (3D) picture.

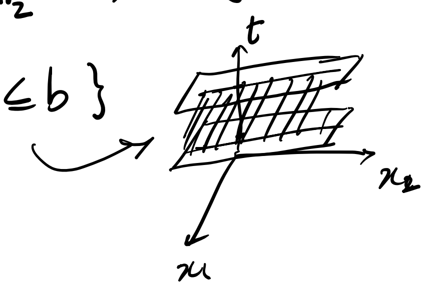


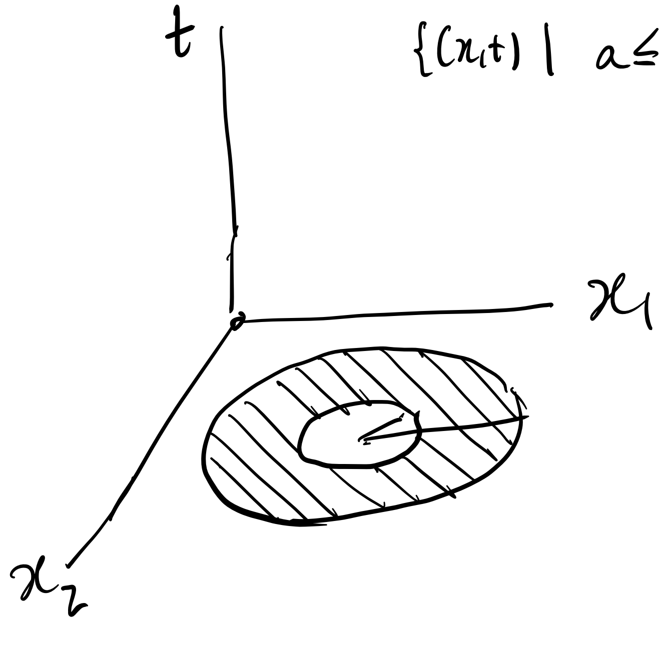
$\{(x,t) \mid \|x-x_0\|_2 \le t$ for all $\underline{a \le t \le b}\}$

convex? yes. intersection of cone

$\{(x,t) \mid \|x-x_0\|_2 \le t,\ t \ge 0\}$ and

$\{(x,t) \mid a \le t \le b\}$





Acknowledgements

- TA : Adhyyan / Grader: Ajay
- Books/classes by Boyd, Vandenberghe, El Ghaoui
- All ee445 students (for trying out this new course with us!)

- Good luck on the final exam!

(please fill in course eval's)