All hw should be uploaded to canvas as a *pdf*. Make sure that if you scan your handwritten notes that they are legible and appropriately oriented. If you use an online resource to solve any problem, please appropriately cite that source.
The material covered in this Extra Credit HW is included in the final exam.

Problem 1. (Counterexamples to convexity/concavity.) The function $f(x) = x_1x_2$ where $x = (x_1, x_2) \in \mathbf{R}^2_+$, is neither convex nor concave (as you show in **HW5, Prob. 4a**). Here you will give counterexamples for convexity and concavity.

a. Give two points $x \in \mathbf{R}^2$ and $y \in \mathbf{R}^2$ for which the following doesn't hold:

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y) \quad \text{for} \quad 0 \le \theta \le 1.$$

b. Give two points $x \in \mathbf{R}^2$ and $y \in \mathbf{R}^2$ for which the following doesn't hold:

$$f(\theta x + (1-\theta)y) \ge \theta f(x) + (1-\theta)f(y) \quad \text{for} \quad 0 \le \theta \le 1.$$

Problem 2. (Weighted least-squares cost as a function of weights.) Let $a_1, \ldots, a_n \in \mathbf{R}^m$. In a weighted least-squares problem (that we saw in **HW3, Prob. 3**) we minimize the objective function

$$\sum_{i=1}^{n} w_i (a_i^T x - b_i)^2$$

over $x \in \mathbf{R}^m$, where w_i are called weights. In this problem, we define the *optimal weighted least squares cost* as

$$g(w) = \min_x \sum_{i=1}^{n} w_i (a_i^T x - b_i)^2,$$

with domain $\mathbf{dom}\, g = \{w \mid g(w) > -\infty\}$. Show that $g(w)$ is concave (i.e., $-g$ is convex) as a function of w.

Problem 3. (Measures of the "spread" in a vector.) For $x \in \mathbf{R}^n$, we define three functions below as measures of the spread or width of its entries. Each of these functions measures the spread of the values of the entries of x; for example, each function is zero if and only if all components of x are equal, and each function is unaffected if a constant is added to each component of x.
Here is the question: For each one, either show that it is convex (as a function of x), or give a counterexample showing it is not convex (specific points x and y for which the convexity inequality fails).

a. The *spread*, defined as

$$\phi_{\text{sprd}}(x) = \max_{i=1,\ldots,n} x_i - \min_{i=1,\ldots,n} x_i.$$

This is the *width* of the smallest interval that contains all the elements of x.

b. The *standard deviation*, defined (recall **Module 1, Lec. 2**) as

$$\phi_{\text{stdev}}(x) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 \right)^{1/2}.$$

(This is also the statistical standard deviation of a random variable that takes the values $x_1, \ldots, x_n$, each with probability $1/n$, but you don't need any statistics to solve this problem).

c. The *average absolute deviation from the median* of the values:

$$\phi_{\text{aamd}}(x) = (1/n) \sum_{i=1}^{n} |x_i - \text{med}(x)|,$$

where $\text{med}(x)$ denotes the median of the entries of x, defined as follows: If $n = 2k - 1$ is odd, then the median is defined as the value of middle entry when the components are sorted, *i.e.*, $\text{med}(x) = x_{[k]}$, the kth largest element among the values $x_1, \ldots, x_n$. If $n = 2k$ is even, we define the median as the average of the two middle values, *i.e.*, $\text{med}(x) = (x_{[k]} + x_{[k+1]})/2$.

Problem 4. (Weighted sum-k-largest function.) Show that $f(x) = \sum_{i=1}^{r} \alpha_i x_{[i]}$ is a convex function of x, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 0$, and $x_{[i]}$ denotes the ith largest component of x.
Hint: You can use the fact that $f(x) = \sum_{i=1}^{k} x_{[i]}$ is convex on $\mathbf{R}^n$ (this was shown in the TA session on 5/27).