All hw should be uploaded to canvas as a *pdf*. Make sure that if you scan your handwritten notes that they are legible and appropriately oriented. If you use an online resource to solve any problem, please appropriately cite that source.

Notes:

a. Homework is always due on Sundays at 11:59PM; however, submitting by Friday at 11:59PM will get you extra credit.

b. This homework (HW0) is graded based on completion only. You should answer all the questions to get full credit. The purpose is to test your background knowledge in linear algebra and calculus (prereqs for this course) to assess how much time we need to spend reviewing relevant concepts. If you do not know how to answer a problem, state that you do not know the answer. You must do this to get credit. It is in your best interest to answer these questions honestly since it will determine the trajectory of the course. Ideally we will not need to spend too much time reviewing material from prereqs and can instead focus on machine learning and optimization relevant foundations.

Note that we use the notation $v = (v_1, v_2, \ldots, v_n)$ and equivalently,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Problem 1. (Inner Product Properties.) Consider the following vectors and scalars:

$$v = (v_1, v_2, \ldots, v_n),\ w = (w_1, w_2, \ldots, w_n),\ u = (u_1, u_2, \ldots, u_n), \qquad \alpha, \gamma$$

Verify the following properties. That is, show that the right-hand side of the expression is equal to the left-hand side.

a. $(\gamma v)^\top w = \gamma(v^\top w)$

b. $(u + v)^\top w = u^\top w + v^\top w$

Problem 2. (Inner Products.) Consider scalars v_i for $i = 1, \ldots, n$.

a. Write $\sum_{i=1}^n v_i^2$ in terms of an inner product of two vectors.

b. Write the average of the scalars as the inner product of two vectors—i.e., $\frac{1}{n}\sum_{i=1}^n v_i$

c. For n even, write $\sum_{i=1}^n v_i$ as the inner product of two vectors where one of the vectors is $w = (v_1, 0, v_2, 0, \ldots, v_n, 0)$. What is the dimension of w?

Problem 3. (Subspaces and dimension.) Recall that a nonempty subset $\mathcal{S}$ of $\mathbb{R}^n$ is a *subspace* if, for any scalars α, β, we have that

$$x, y \in \mathcal{S} \implies \alpha x + \beta y \in \mathcal{S}.$$

Consider the set $\mathcal{S}$ of points in $\mathbb{R}^3$ such that

$$\mathcal{S} = \{(x_1, x_2, x_3) \mid x_1 + 2x_2 + 3x_3 = 0,\ 3x_1 + 2x_2 + x_3 = 0\}$$

Show that $\mathcal{S}$ is a subspace of $\mathbf{R}^3$. What is the dimension of $\mathcal{S}$?

Problem 4. (Gradients and Hessians.) Consider a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. That is, $f(x) = y$ (or equivalently, $f : x \mapsto y$—where $x \in \mathbb{R}^n$ is an n-vector and $y \in \mathbb{R}$ is a scalar.

1. Write out an expression for the gradient of f and the Hessian of f—i.e., $\nabla f(x)$ and $\nabla^2 f(x)$ respectively.
2. Fine the gradient and the Hessian for the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$, $f(x_1, x_2) = x_1 x_2^2$.

Problem 5. (Range and Null Space.) For a matrix A, recall that the null space (or equivalently kernel) is defined by

$$\ker(A) = \mathrm{null}(A) = \{x \mid\ Ax = 0\}$$

and the range space or image or column space of A is defined by

$$\mathrm{im}(A) = \mathrm{range}(A) = \{y \mid \exists\ x \text{ such that } Ax = y\},$$

Note that $\exists$="there exists". Consider the matrix A and two vectors v, w defined by

$$A = \begin{bmatrix} 1 & 3 & 0 & 7 \\ 2 & 6 & 5 & 9 \\ 3 & 9 & 5 & 16 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \ w = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

a. Is v in the null space of A?

b. Find a basis for the null space of A.

c. Is w in the range of A?

d. Find a basis for the range space of A.

Problem 6. (Linear Maps.) Recall that a map $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation if $T(u+v) = T(u) + T(v)$ and $T(cu) = cT(u)$ for vectors u, v, w and scalar c. Consider the map $T : \mathbb{R}^4 \to \mathbb{R}^4$ that takes a vector in $\mathbb{R}^4$ and reverses its entries—i.e.,

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \mapsto \begin{bmatrix} v_4 \\ v_3 \\ v_2 \\ v_1 \end{bmatrix}$$

a. Show this map is a linear transformation.

b. Find a matrix representation of this map—i.e., find a matrix A such that $T(u) = Au$. **Hint**: use the standard basis vectors and apply T to each of those.

Problem 7. (Matrix-vector multiplication.) Suppose x is an n-vector and A an $n \times n$ matrix. Write the expression $x^\top Ax$ in terms of the vector entries x_i and matrix entries A_{ij}. Does the expression simplify if the matrix A is diagonal? How?

Problem 8. (Eigenvalues and eigenvectors.) Recall that the unit-length vector v is an eigenvector of a matrix A if there exists a scalar λ such that $Av = \lambda v$. Find the eigenvector corresponding to the eigenvalue $\lambda = 1$ for the *shift* matrix given below,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$